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ORIGINAL PAPER

Simulations of two dimensional hopper flow

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Abstract We report on discrete element simulations of the flow out of a hopper. The geometry of the setup, as well as the material parameters, are taken directly from the experiments reported by Tang and Behringer (Chaos 21:041107, 2011; IFRPI-NSF collaboratory: 2D hopper data summary, 2010). The simulations consider flow of elastic, frictional, disk-like particles as the slope of the hopper walls and the size of the opening are varied, so that we can consider both the regime where jamming happens frequently (small openings) and where it occurs very rarely, or not at all (large openings), as well as the influence of the slope of hopper walls. We discuss the distribution of jamming events, the mass flux out of the hopper, and the influence of material parameters on these quantities. In addition, we consider velocity and pressure fields in the hopper, as well as their fluctuations.

Keywords Hopper flow · Discrete element simulations · Jamming

1 Introduction

The flow of granular matter out of a hopper is one of the classical problems and as such has been studied for a long time. Despite a number of extensive studies, some of rather basic questions are still waiting for answers. For example, it has been considered for many years that the flux of granular particles through an opening is well described by Beverloo equation [1] which predicts that (in two spatial dimensions), the mass flux is related to the ratio of the opening of the hopper and average particle size by the $3/2$ law. However, recent results [2,3] suggest that this universal behavior may not be satisfied over a large range of relevant parameters. Other questions come to mind, such that whether jamming is (at least in principle) possible for large hopper openings. The recent experimental work in three spatial dimension (3D) [4], suggests existence of a critical opening size, such that jamming is not possible for sufficiently large hopper openings. However, it is not clear whether these results hold in two spatial dimensions (2D), and if they do not, what is the mechanism responsible for such a dramatic influence of the number of physical dimensions on jamming of a hopper.

Due to the complexity of the problem considered, simulations appear as a very appropriate tool to be used in its analysis, in particular since they allow for a relatively simple variation of material parameters and hopper geometries. However, simulations are by necessity based on simplified models, and it is not always clear that the implemented simplifications are appropriate. Therefore, it appears reasonable to carry out joint experimental/computational study such that the needed variables and material parameters can be provided directly by experiments (to the degree possible), and the results of simulations can be compared to the corresponding experiments. This paper presents partial results of such an effort; the parameters that are used in the simulations are taken from the experiments carried out in Behringer's lab [5,6], but the focus of the present work is on the simulations results alone, in particular in the context of some of the open questions mentioned briefly above. Future work will include a direct comparison between the experimental and theoretical results.

This paper is organized as follows. In Sect. 2 we give an outline of the setup of the simulations. Then, we proceed

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with presenting and discussing the results in Sect. 3, with particular emphasis on differentiating between jamming and non-jamming regimes, and discussing the issues related to statistical nature of the results. Section 4 is devoted to a brief discussion and summary.

2 Setup of the simulations

In our simulations, we attempt to model as closely as possible a typical 2D experimental setup, used in [5,6]. In the experiments, the disk-like particles are poured in a hopper consisting of front and back walls, which are placed slightly over one-particle-size apart. After particles are placed in the hopper, the outlet is opened, and the particles flow through under gravity. In simulations, we attempt to reproduce as closely as possible the experimental flow geometry. Figure 1 shows the setup and the parameters that define the hopper's geometry. The width of the hopper is 43 cm, and size of the outlet and slope of the hopper walls can be controlled. There is a total of 8,750 particles, which are bidisperse disks, 62 % of which are of diameter 0.602 cm, and the rest is of diameter 0.770 cm. The width of the particles is 0.32 cm. The material properties of the particles are as follows: $\rho = 10^3$ kg/m^3; Young's modulus $Y = 4.8 \times 10^6$ Pa; Poisson ratio $\sigma = 0.5$; Coefficient of static friction $\mu = 0.7{-}0.8$. In simulations, the hopper walls are made of (immobile) particles; this implementation allows to easily simulate the walls of different roughness, if so desired.

The implemented simulation model assumes that the particles are inelastic frictional compressible disks interacting only when in contact. The interaction is based on Cundall–Strack type of model [7]. The equations of motion that are solved for each particle are as follows

$$m_i \frac{d^2\mathbf{r}_i}{dt^2} = \mathbf{F}^n_{i,j} + m_i \mathbf{g},$$
$$I_i \frac{d\boldsymbol{\omega}_i}{dt} = -\frac{1}{2} d_i \mathbf{n} \times \mathbf{F}^t_{i,j}, \qquad (1)$$

where $\mathbf{r}_i$, ω_i, m_i, d_i, I_i are the position, angular velocity, mass, diameter, and moment of inertia of the ith particle, respectively, and $\mathbf{g}$ is the acceleration of gravity. The normal force between the particles i and j is given by

$$\mathbf{F}^n_{i,j} = \left[k_n x - \gamma_n \bar{m} \mathbf{v}_{i,j}\right] \mathbf{n}, \qquad (2)$$

where the normal direction is defined by $\mathbf{n} = \mathbf{r}_{i,j}/r_{i,j}$, $r_{i,j} = |\mathbf{r}_{i,j}|$, $\mathbf{r}_{i,j} = \mathbf{r}_i - \mathbf{r}_j$. The compression is defined by $x = d_{ave} - r_{i,j}$, where $d_{ave} = (d_i + d_j)/2$; $\mathbf{v}^n_{i,j}$ is the relative normal velocity, and $\bar{m} = m_i m_j/(m_i + m_j)$ is the reduced mass. We use linear force model ($|\mathbf{F}^n| \propto x$) as commonly used for 2D particles; k_n is the spring constant discussed further below.

In our attempt to keep the results closely related to the experimental ones, in this paper we will use dimensional quantities. In the following we will however need a typical time scale relevant for the binary collisions between the particles, defined as

$$\tau_c = \pi \sqrt{m/(2k_n)} \qquad (3)$$

where m is taken as the mass of large particles. The coefficient of restitution, e_n, is related to the damping coefficient γ_n by $\gamma_n = -2 \ln e_n/\tau_c$, see, e.g., [8]. We take e_n to be a constant and ignore its possible velocity dependence, discussed, e.g., in [9].

The tangential force is based on Cundall–Strack type of model [7], where a tangential spring of zero length is introduced as a new contact between two particles forms at $t = t_0$. Due to relative motion of the particles, the spring length, $\boldsymbol{\xi}$ evolves as

$$\boldsymbol{\xi} = \int_{t_0}^{t} \mathbf{v}^t_{i,j}(t')\, dt',$$

where $\mathbf{v}^t_{i,j} = \mathbf{v}_{i,j} - \mathbf{v}^n_{i,j}$. For long lasting contacts, $\boldsymbol{\xi}$ may not remain parallel to the current tangential direction defined by $\mathbf{t} = \mathbf{v}^t_{i,j}/|\mathbf{v}^t_{i,j}|$ (see, e.g., [10,11]); we therefore define corrected $\boldsymbol{\xi}' = \boldsymbol{\xi} - \mathbf{n}(\mathbf{n} \cdot \boldsymbol{\xi})$ and introduce the test force

$$\mathbf{F}^{t*} = -k_t \boldsymbol{\xi}' - \gamma_t \mathbf{v}^t_{i,j},$$

where γ_t is the coefficient of viscous damping in the tangential direction (we use $\gamma_t = \gamma_n/2$). To keep the magnitude of

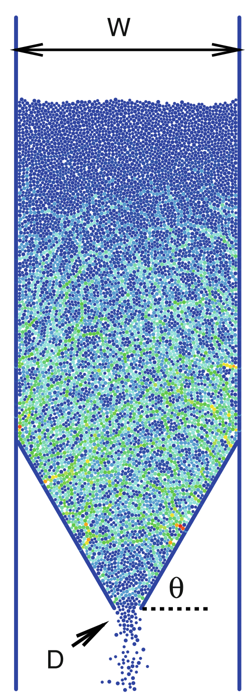

Fig. 1 Hopper setup. The particles in the hopper are colored according to the normal force experienced. Gravitational compaction leads to larger forces in the lower parts of the hopper

tangential force smaller than the Coulomb threshold, specified by $\mu\mathbf{F}^n$, where μ is the coefficient of static friction, we define the tangential force by

$$\mathbf{F}^t = min(\mu|\mathbf{F}^n|, |\mathbf{F}^{t*}|)\frac{\mathbf{F}^{t*}}{|\mathbf{F}^{t*}|}. \quad (4)$$

In addition, $\boldsymbol{\xi}'$ is reduced to the length corresponding to the value of $|\mathbf{F}^t|$ as needed. This is a commonly used model for static friction, for non-zero k_t. To be able to isolate the effect of static friction, we also consider a kinetic friction model based on viscous damping, which is obtained simply by putting $k_t = 0$. Therefore, depending on whether static friction effects are considered or not we use either $k_t = 0.8k_n$ (the value suggested in [12]), or $k_t = 0.0$. The exact value given to k_t does not seem to be of relevance in the present context. In general, the particles making up the walls are characterized by their own, in general different coefficient of friction and restitution coefficient.

The parameters entering the force model are related to the physical properties of the material as follows. The coefficient of damping in the normal direction uses coefficient of restitution as specified above. Then, the spring constant, k_n, is connected to the material properties (Young modulus, Poisson ratio) using the method described e.g., in [8], outlined briefly here. The method is based on equating the collision time resulting from the linear model, τ_c, with the one resulting from general non-linear model for interaction of two spherical particles

$$t_{col} = I(\beta)\left[1+\frac{\beta}{2}\right]^{1/(2+\beta)}\left[\frac{m}{Ed_l^{1-\beta}}\right]^{1/(2+\beta)} v_0^{-\beta/(2+\beta)}, \quad (5)$$

where $E = Y/[3(1-\sigma^2)]$. We use $\beta = 1/2$ as appropriate for Hertzian interaction model; d_l is the diameter of a large particle. Another parameter entering this (nonlinear) model is relative velocity, v_0, of the interacting particle. While this quantity is not precisely known (it varies depending on the flow), we note that t_{col} depends only weakly on v_0, $t_{col} \propto v_0^{-1/5}$. We chose as appropriate value for v_0 the speed that a particle would have after falling from rest under gravity a distance of its own diameter. Then, using the experimentally given Y and σ, we find for large particles $t_{col} \approx 8.0 \times 10^{-4}$ sec. Applying the same approach to small particles gives a similar, slightly smaller, value. Putting $t_{col} = \tau_c$ and using Eq. (3) above, we then extract the value of k_n that is used in the simulations.

Using the interaction model described above, we carry out the simulations using 4th order predictor–corrector model [13]. The time step, Δt, is chosen short compared to τ_c, $\Delta t = \tau_c/\delta$, $\delta = 50$. It has been verified that the results are fully converged and independent of the time step size.

Initially, given the outlet size, D, and the angle, θ, of the inclined walls with the horizontal, particles are placed in the area above the inclined walls on a square lattice. Each particle's diameter is chosen randomly. Then, the particles are left to settle under gravity (with hopper outlet closed). The obtained configuration is used as an initial condition for the simulations of the flow out of the hopper. As discussed later in the text, we have found that in some cases it is necessary to carry out multiple realizations of the flow process in order to reduce statistical noise; these multiple realizations are implemented by modifying the initial configuration of the particles.

The simulations carried out with smaller size of the outlet, D, lead to particles that jam at the outlet, as it is commonly observed experimentally. To unjam the hopper, we shake the whole system—similarly as in the experiments where a controlled tap is applied [14]. To monitor for possible jamming, in the simulations we keep track of the number of particles in the area just below the hopper outlet, of the height equal to $2d_l$. If no particles are found in this region during the time $t_{check} = 2\sqrt{d_l/g}$, the flow is assumed to be jammed; here g is the acceleration of gravity. This value of t_{check} is found to be sufficiently long to ensure that the system is indeed jammed. On the other hand, it is sufficiently short so that it does not lead to a large inaccuracy of the survival time, defined as the time between two consecutive jams/shakes. This inaccuracy results from the fact that during t_{check} it is not known whether the system is jammed or not. The error of estimating a survival time using this approach is found to be of the order of 0.1 s. The shake itself consist of a single sinusoidal oscillation of the amplitude and duration chosen in such a way that it is (almost always) sufficient to unjam the system, without disturbing it significantly. In rare occasions when a single oscillation is not sufficient to unjam the system, consecutive oscillation(s) are applied. In what follows, we will discuss the survival time, and the empty time, defined as the time needed to empty the whole hopper, excluding the shaking time, if any.

3 Results

We start by discussing in Sect. 3.1 the results of single realization simulations, where we vary some of the parameters entering the model. We concentrate in particular on the influence of the coefficient of restitution, e_n and the friction coefficient, μ. Then, we proceed in Sect. 3.2 by discussing the results of multiple realizations (typically 40 realizations are used) that were carried out in order to improve statistical interpretation of the results. Due to the computational cost involved, these multiple realization simulations are considered only for a subset of the parameters considered. In Sect. 3.3, we discuss the features of the velocity and pressure fields that develop during hopper discharge.

3.1 Single realization results

Tables 1 and 2 show the results for the mean survival time for given D and θ, and for a set of values of e_n and μ for both granular particles and the walls. Table 1 concentrates on the case where the flow jams frequently (small outlet); the Table 2 considers the non-jamming case (large outlet).

The conclusions of the single realization results are as follows:

- The survival times are characterized by a large variation, and there is no clear influence of the material parameters.

Table 1 Mean survival time (in seconds) for varied system parameters of outlet size $D = 2.7$ cm (approximately 4 average particle diameters) and angle $\theta = 45°$ (single realization results)

e_n	Wall e_n	System μ	Wall μ	Survival time
0.5	0.5	0.5	0.8	6.20
0.5	0.5	0.6	0.8	5.60
0.5	0.5	0.7	0.8	2.11
0.5	0.5	0.8	0.8	2.76
0.8	0.5	0.5	0.8	6.14
0.8	0.5	0.6	0.8	5.82
0.8	0.5	0.7	0.8	3.03
0.8	0.5	0.8	0.8	16.54
0.5	0.3	0.8	0.8	8.47
0.5	0.8	0.8	0.8	5.24
0.5	0.5	0.8	0.5	4.55
0.3	0.5	0.8	0.8	2.89
0.5	0.5	0.8	0.8	14.46
0.5	0.5	0.8	0.8	4.43
0.5	0.5	0.8	0.8	5.56

Large variation of the survival times

Table 2 Empty time (in seconds) for varied system parameters and for the outlet size, $D = 4.5$ cm (approximately 6.75 average particle diameters) and angle, $\theta = 60°$ (single realization results)

System e_n	System μ	Empty time
0.5	0.5	12.16
0.5	0.6	12.54
0.5	0.7	13.07
0.5	0.8	13.23
0.8	0.5	12.17
0.8	0.6	12.46
0.8	0.7	12.95
0.8	0.8	13.22

Here we use $e_n = 0.8$ and $\mu = 0.5$ for the wall particles

  We note that additional simulations carried out without static friction, $\mu = 0$, or with completely elastic particles, $e_n = 1.0$, do not lead to jamming, showing the importance of both static friction and inelasticity.
- The empty times appear to vary very weakly, if at all, for the considered variation of the model parameters.

Table 1 suggests that a single realization is not sufficient to obtain statistically relevant results for the survival times. For this reason, we have carried out multiple (40) realizations in order to improve the statistics. Due to the computational expense involved, we have carried out these simulations only for one set of parameters: $e_n = 0.5$, $\mu = 0.8$ (for both the system particles and the walls), and $\theta = 45°$.

3.2 Multiple realization results

Table 3 gives the algebraic means of the survival times averaged over 40 realizations, together with the total number of jamming events. As expected, we find a strong increase of the mean survival time as the size of the hopper opening is increased. We have not found any jamming event for $D > 3.5$ cm, corresponding to 5–6 average particle diameters.

In addition to the strong influence of D, we also observe a strong influence of the hopper opening angle, θ: larger θ's (steeper hopper walls) lead most of the time to significantly longer mean survival times. This finding could be explained based on the expected larger flow rates for large angles, which should decrease the probability of jamming.

In general, one may expect that the distribution of survival times obeys Poisson process, that is, the probability that the flow will survive for time t without being jammed is given by $P(t) \propto \exp(-t/\tau)$; see also [15] for much more detailed discussion of this assumption, and [16] for a discussion based

Table 3 The mean survival time (in seconds) (algebraic means over 40 realizations) and the total number of jamming events

Outlet	Mean survival time	No. jamming events
$D = 2.7$ cm, $\theta = 15°$	3.06	524
$D = 2.7$ cm, $\theta = 30°$	3.53	371
$D = 2.7$ cm, $\theta = 45°$	4.50	261
$D = 2.7$ cm, $\theta = 60°$	6.55	98
$D = 2.9$ cm, $\theta = 15°$	5.58	228
$D = 2.9$ cm, $\theta = 30°$	6.62	146
$D = 2.9$ cm, $\theta = 45°$	6.45	124
$D = 2.9$ cm, $\theta = 60°$	10.10	47
$D = 3.1$ cm, $\theta = 45°$	8.00	48
$D = 3.3$ cm, $\theta = 45°$	10.89	26
$D = 3.5$ cm, $\theta = 45°$	12.53	8

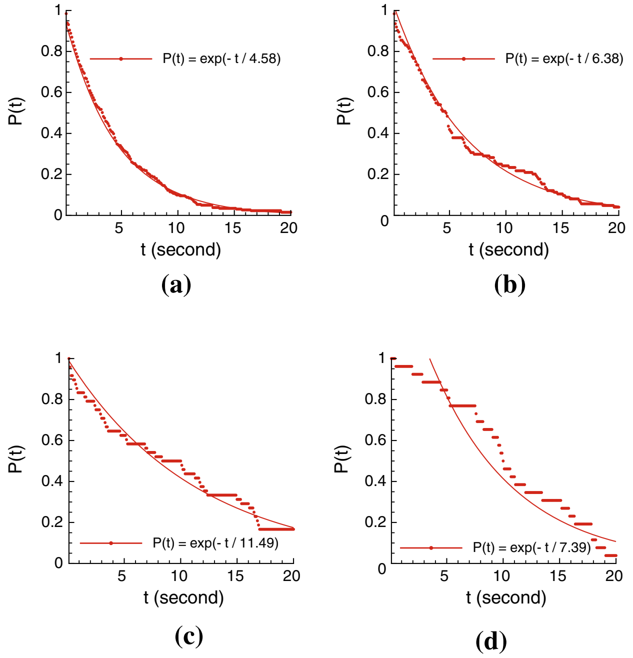

Fig. 2 Distribution of the survival times for $D = 2.7$ cm and $\theta = 45°$. The data are obtained from 40 realizations. **a** $D = 2.7$, $\theta = 45°$, $\chi^2 = 0.9970$. **b** $D = 2.9$, $\theta = 45°$, $\chi^2 = 0.9806$. **c** $D = 3.1$, $\theta = 45°$, $\chi^2 = 0.9502$. **d** $D = 3.3$, $\theta = 45°$, $\chi^2 = 0.8373$

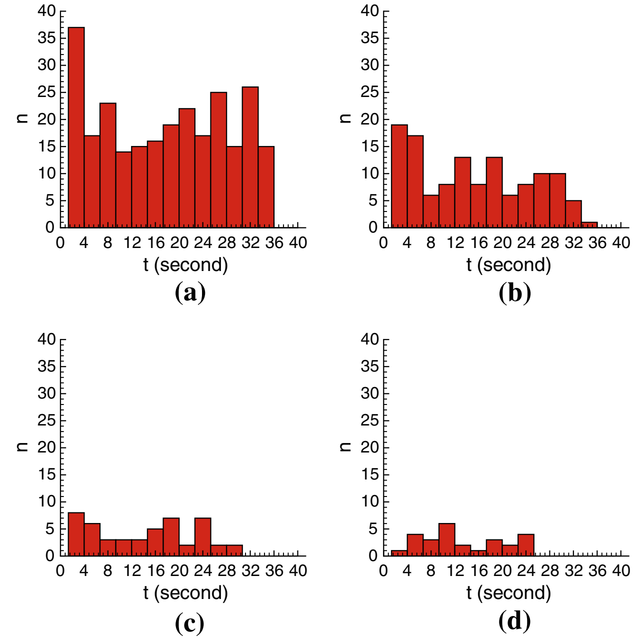

Fig. 3 Distribution of survival times as function of elapsed time from the beginning of the simulations. Here, $\theta = 45°$. The data are obtained from 40 realizations. **a** $D = 2.7$ cm. **b** $D = 2.9$ cm. **c** $D = 3.1$ cm. **d** $D = 3.3$ cm

on the approach originating from the percolation theory. Figure 2 shows the distribution of survival times for few values of D and for $\theta = 45°$. For the small D's, shown in Fig. 2a, b, we find very good exponential fit to the distribution of survival times, consistently with the experiments [6, 16]. For the larger outlet sizes (Fig. 2c, d), the quality of the exponential fit deteriorates, as confirmed by considering χ^2 values. In particular, the fact that the exponential fit gives larger mean survival time for $D = 3.1$ cm than for $D = 3.3$ cm illustrates the difficulties experienced for larger D's. Therefore, we conclude that, while for smaller D's, exponential fit of the distribution of the survival times is reasonably accurate, for large D's it is difficult to decide, based on the presented results, whether exponential behavior holds. A similar conclusion was reached in the two-dimensional experiments carried out with monodisperse particles [15].

Next question to ask is whether their exists a critical value of the hopper opening, D_c, such that for $D > D_c$, jamming is not possible, as suggested recently [4]. Considering the mean survival times based on the algebraic means given in Table 3, we find a distribution which appears consistent with the exponential one: if this finding is indeed accurate, the prediction is that probability of jamming decreases exponentially with D and therefore such a critical D_c would not exists. However, based on our results it is not possible to reach a definite conclusion: note that for $D = 3.5$ cm, we find only eight jamming events in 40 realizations, and therefore a huge number of realizations (computational or experimental) may be needed to answer this question. We note that the experimental results [6] are also consisted with the exponential dependence of the jamming probability on the hopper opening, albeit with a different coefficient in the exponent; the reasons for these discrepancy are still to be understood. In any case, we note that our results for the survival times obtained for the hoppers characterized by different θ's suggest that the behavior for large D's may easily depend on the opening angle itself.

Next, we discuss whether there is any evidence that survival times may depend on the amount of material remaining in the hopper. Figure 3 shows the distribution of the number of times jamming occurred as a function of elapsed time for four opening sizes. Within the statistical fluctuations, the jamming times appear to be independent of the elapsed time and therefore on the amount of material in the hopper.

The simulation results can be further used to discuss more precisely the time needed to empty the whole hopper and the corresponding mass flux. Here we consider both small ($D \le 3.5$ cm) and large ($D > 3.5$ cm) outlet sizes. The results are shown in the context of Beverloo equation [1] which predicts the mass flow rate of the following form (in 2D)

$$V = C\rho_b\sqrt{g}(D - kd)^{3/2}. \tag{6}$$

Here V is the mass flow rate through the outlet, ρ_b is the bulk density, g is the acceleration of gravity, and d is the

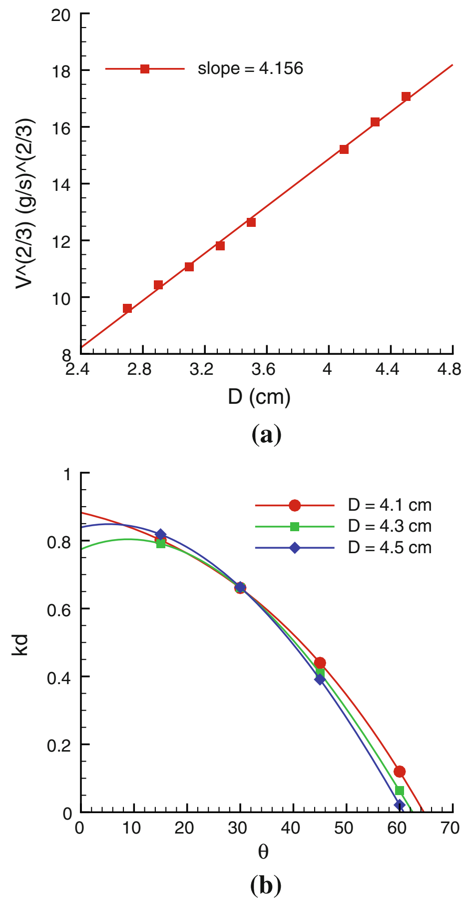

Fig. 4 Comparison of simulation results to the Beverloo equation. **a** Mass flux for $\theta = 45°$. **b** Dependence of the constant, k, see Eq. (6), on the outlet angle, θ

Table 4 Mean empty time data. Averages over 40 realizations are used for $D \leq 3.5$ cm, and single realizations are reported for larger D's since if there is no jamming, the fluctuations between realizations are negligible

Outlet	$\theta = 45°$
$D = 2.7$ cm	34.45
$D = 2.9$ cm	30.44
$D = 3.1$ cm	27.83
$D = 3.3$ cm	25.25
$D = 3.5$ cm	22.82
$D = 4.1$ cm	17.27
$D = 4.3$ cm	15.76
$D = 4.5$ cm	14.52

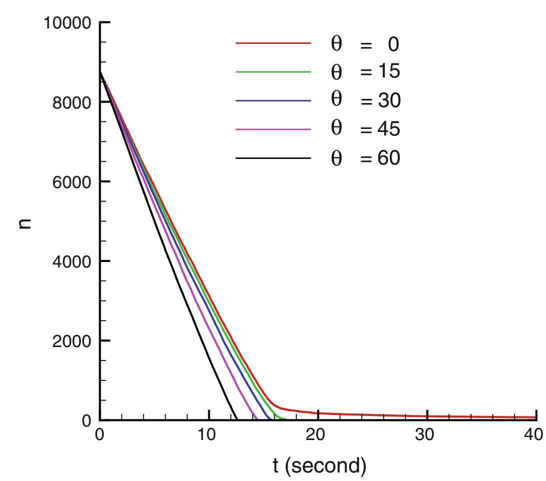

Fig. 5 Number of particles in the hopper as a function of time for different hopper angles ($D = 4.5$ cm)

mean diameter of the particles in the hopper. C and k are fitting coefficients. The value of C may depend on the friction coefficient, and k generally depends on the particle shape and on the opening angle. The validity of the Beverloo law has been tested for $D \gg d$ so that no jamming takes place. Here we check the validity of the proposed scaling for both small and large D's.

Figure 4 shows the results for the flux, V, for different D's. The part (a) shows that in the considered range, the scaling predicted by the Beverloo formulation is satisfied for both small and large outlets. We note that recent experiments in two dimensions have found deviations from Beverloo-like behavior [3]; however, in our simulations we do not find any systematic deviation, at least for the parameters considered. Further work will be needed to address this difference. The part (b) of this figure shows that the constant k in the Beverloo equation indeed strongly depends on the hopper angle, θ, and it strongly decreases for large θ, as expected based on the geometric considerations. Table 4 summarizes the results for empty times for the considered outlet sizes.

Figure 5 shows the number of particles in the hopper as a function of time, for large $D = 4.5$ cm, and for few different θ's. We see that the mass flux (the rate at which the number of particles decreases) is essentially constant, during any given simulation, except close to the end of a simulation and for small outlet angles. In particular we note that we do not find any relevant dependence of the flux on the remaining amount of material in the hopper. Note also that the hopper with $\theta = 0°$ never empties since some particles remain at the bottom wall.

3.3 Velocities and pressures

We proceed to briefly discuss the results for the velocities and pressures in the hopper, with the main goal of illustrating the influence of the hopper opening angle on these quantities, and furthermore showing that even for large outlets (so that jamming does not occur), large fluctuations of the velocities and pressure are present. The results are obtained by first time-averaging over the interval of $1/100$ s, and then carrying out the spatial average over the cells of size 2×2 (in units of d_l). The fluctuation plots show the standard deviation of the time averages.

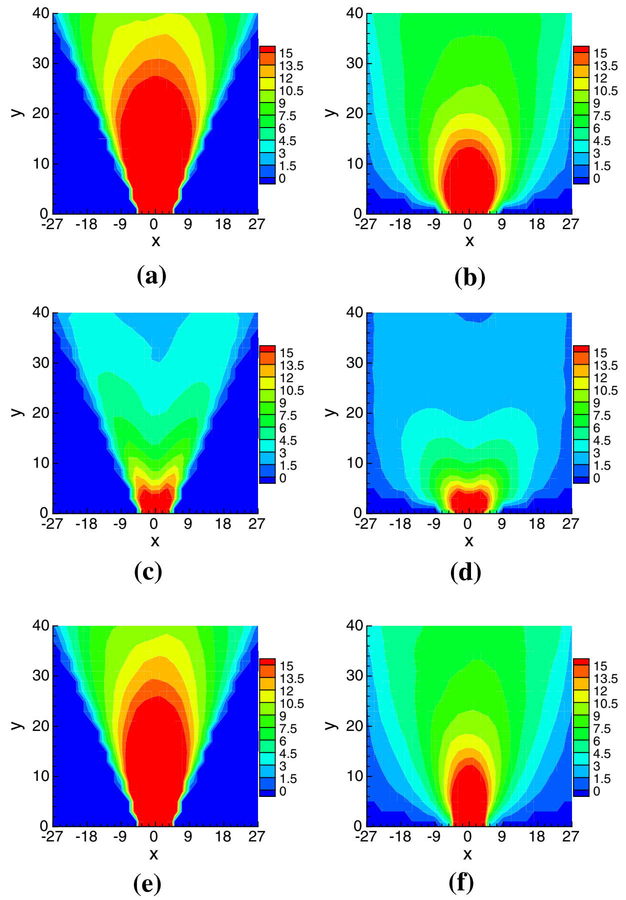

Fig. 6 Absolute values of the averaged velocities (in cm/s) of the particles for $D = 4.5$ and two θ's. Slight waviness next to the hopper walls is due to the cell-averaging procedure implemented. **a** Velocity magnitude, $\theta = 60°$. **b** Velocity magnitude, $\theta = 15°$. **c** x-component of velocity, $\theta = 60°$. **d** x-component of velocity, $\theta = 15°$. **e** y-component of velocity, $\theta = 60°$. f y-component of velocity, $\theta = 15°$

Figure 6 shows contour plots of the averaged velocities for two values of θ. We find the results that are qualitatively similar to the ones reported in the experiments [5]. To illustrate the degree of fluctuations of velocities, we also show standard deviations of the velocity field in Fig. 7 for $\theta = 60°$. Note that these deviations are significant, both close to the outlet and further up in the hopper.

Figures 8 and 9 show the pressures (more precisely, the normal force which the particles experience, in units of mg), averaged over space and time in the same manner as the velocities above. We observe strong influence of the opening angle on the magnitude of pressure, with smaller opening angles being characterized by much larger pressures. We expect that further analysis of the pressures and velocities fields, in the regime where jamming occurs, will provide further insight regarding jamming itself and help in understanding of the hopper discharge in that regime as well.

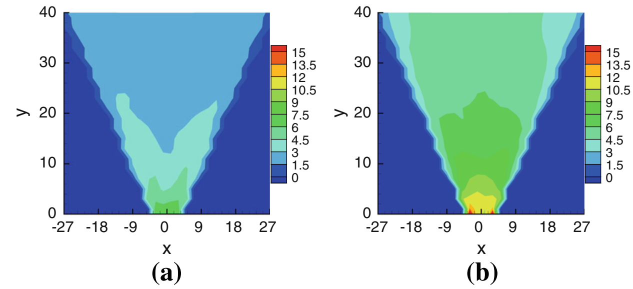

Fig. 7 The fluctuations of the velocity components, given in cm/s. ($D = 4.5$ cm, $\theta = 60°$). **a** x-component of velocity. **b** y-component of velocity

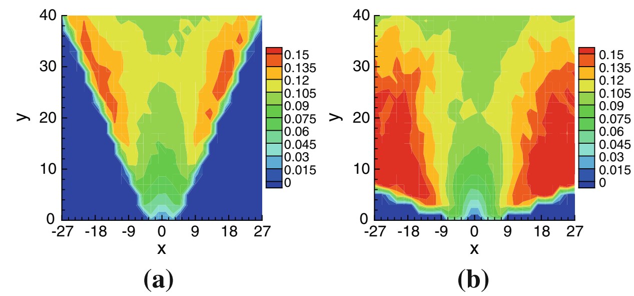

Fig. 8 Averaged pressure fields for $D = 4.5$ cm, and two θ's. **a** $\theta = 60°$. **b** $\theta = 15°$

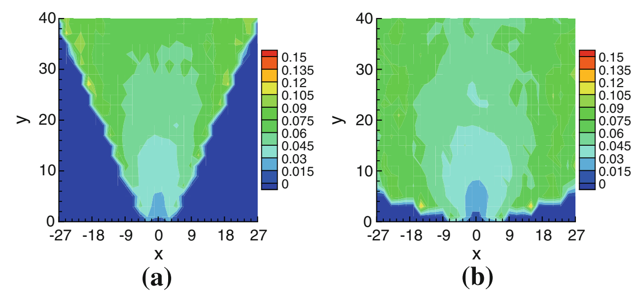

Fig. 9 Fluctuations of pressure fields for $D = 4.5$ cm and two θ's. **a** $\theta = 60°$. **b** $\theta = 15°$

4 Summary

In this work, we have considered the hopper flow by the means of discrete element simulations. When the hoppers with large opening size, D (relative to the particle size), are considered, we find only very weak (if any) influence of the material parameters (friction, coefficient of restitution) on the times needed to empty the whole hopper. For small D's, the particles jam close to the opening, with increased probability of jamming for smaller D's. In this regime, the material parameters become important, since, e.g., the particles whose interaction is modeled by kinetic friction (viscous dissipation), or completely elastic particles, do not jam. For these small D's, and for the particle properties that lead to jamming, the distribution of survival times (the time intervals between consecutive jamming events) is found to vary significantly, and it is necessary to carry a large number of independent realizations in order to obtain statistically meaningful results. The averaged results obtained from these realizations show that the distribution of survival times for small openings follows closely an exponential law, consistently with the Poisson distribution. The jamming probability itself appears to be independent of the number of particles remaining in the hopper, or at least any variation of this probability is within the statistical variation of our results. When larger openings are considered, such that jamming is possible but not probable, the quality of the fit to an exponential law is rather weak, suggesting that very large number of realizations would be needed to either prove or disprove the proposed exponential law. In the case of even larger openings, jamming does not occur, and the outflow of particles is continuous. For the considered bidisperse distribution of particle sizes, the critical value of the hopper opening, beyond which jamming does not occur, at least for the number or realizations carried out, is found to be between 5 and 6. Both small and large openings lead to the values of the mass flux that are in good agreement with the predictions based on Beverloo's formulation.

We note that the model on which our simulations are based is rather simple, and many improvements are possible. For example, we consider linear normal force model; one could ask what is the influence of nonlinearity, if any; the coefficient of restitution is treated as a constant (velocity independent); rolling friction is not considered. Furthermore, we have considered circular particles in two spatial dimensions; it will be of interest to consider the influence of spatial dimensionality and/or of the particle shape on the results presented here.

Preliminary comparison of the current results with the experiments [5,6] that in part inspired this work suggests that the results of simulations are to a large degree consistent with the experimental ones, although there are some differences as well, in particular regarding the dependence of the survival times on the hopper opening size. A close coordination of further experimental, modeling, and computational efforts will be needed to answer the remaining questions. This will be a subject of our future work.

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