Thin Film Flow in a Funnel

Theory/Numerics

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Outline

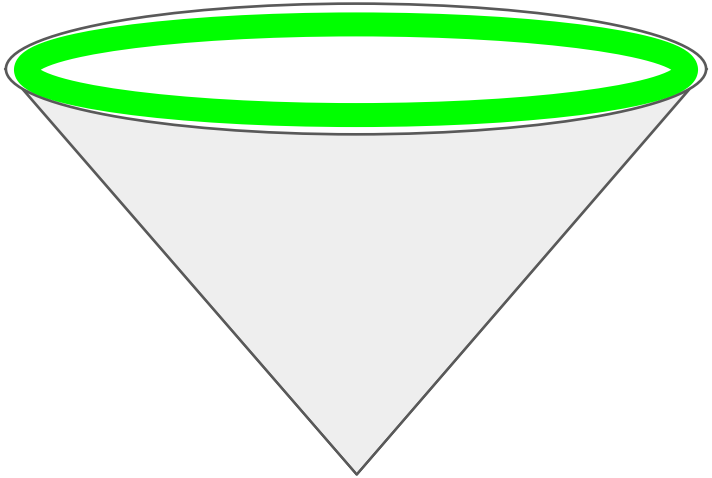
1. Theory

1. Numerics

1. Results and Comparison to Experiments

Theory

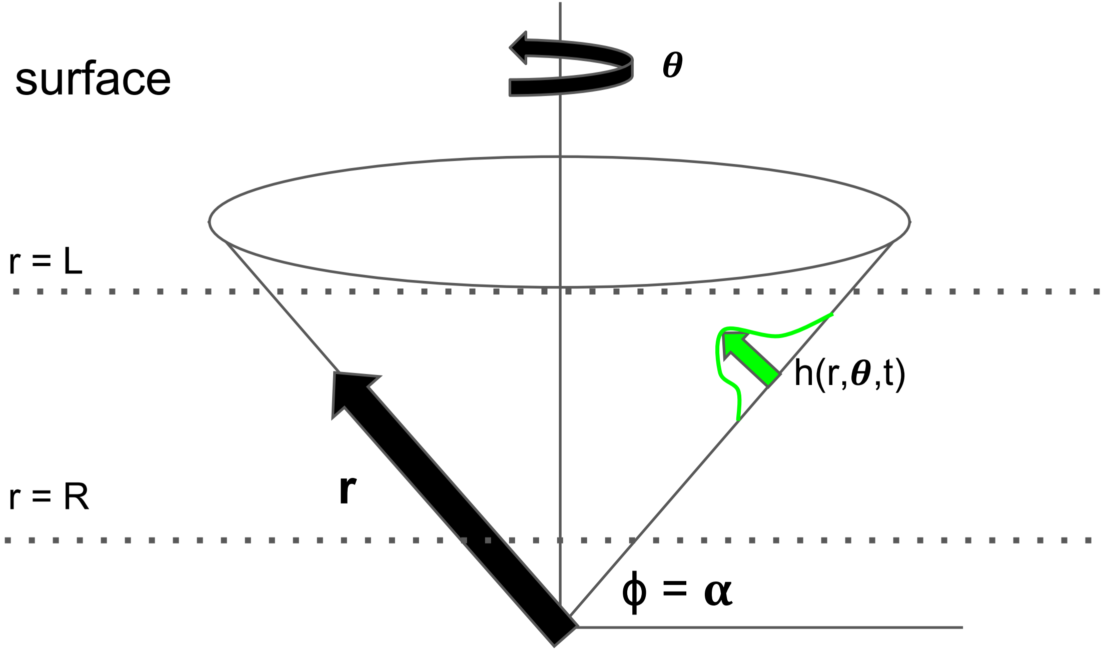
Funnel Geometry



What equation models the height of the fluid as it falls?

Funnel Geometry

Parametrize the surface
(spherical)



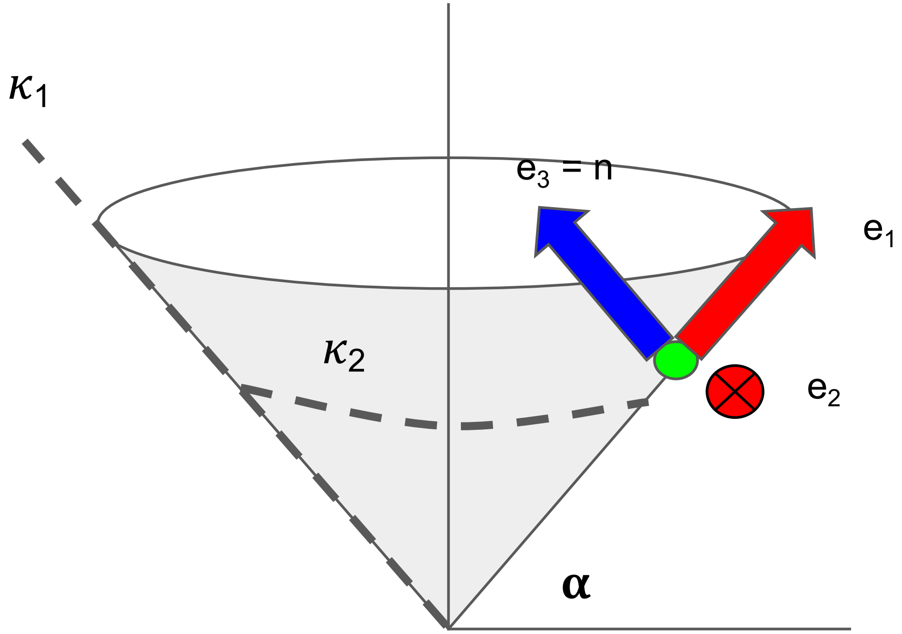
$$(x, y, z) = (r \cos\alpha \cos\theta, r \cos\alpha \sin\theta, r \sin\alpha)$$

Funnel Geometry

Curvatures

$\kappa_1 = 0$

$\kappa_2 = \tan\alpha / r$



Unit vectors

$$\boldsymbol{e}_1 = (\cos\alpha\cos\theta, \cos\alpha\sin\theta, \sin\alpha), \quad \boldsymbol{e}_2 = (-\sin\theta, \cos\theta, 0), \quad \boldsymbol{n} = (-\sin\alpha\cos\theta, -\sin\alpha\sin\theta, \cos\alpha),$$

The Model

Assumptions:

- Very viscous Newtonian incompressible Fluid: dynamics are slow
- Flow is driven by surface tension and gravity.
- The film is thin. The height of the fluid h is much smaller than the rest of scales in the problem. (domain size L-R and radius of curvature r/tan(**α**))

Derivation is performed in **[1]** for an arbitrary stationary smooth curved substrate:

- Start with N-S equations, use assumptions and perform multiple scale analysis.
- The substrate geometry is characterized by a specific set of orthonormal vectors and the principal curvatures.
- A differential equation for time evolution of h is obtained.

Chosen scales:

$$h = a\,\bar{h}, \quad r = a\,\bar{r}, \quad t = t_c\,\bar{t}, \qquad a = \sqrt{\gamma/\rho g} \qquad t_c = 3\mu a/\gamma$$

[1] R.V. Roy, A.J. Roberts and M.E. Simpson. A lubrication model for coating flows over a curved substrate in space.

The Equations

$$h_t = -\nabla_s \cdot \Big[\underbrace{h^3 \nabla_s \Big(\nabla_s^2 h}_{\text{Surface tension}} + \underbrace{\frac{\tan\alpha}{r}}_{\text{Curvature}} \Big) \underbrace{- G \left(\sin\alpha h^3 \boldsymbol{e}_1 + \cos\alpha h^3 \nabla_s h \right) \Big]}_{\text{Gravity}}$$

$\nabla_s \cdot$, ∇_s , ∇_s^2 are divergence, gradient and Laplacian in spherical coords with $\phi = \alpha$

The Equations

To perform LSA, assume solution of the form

$$h(t, r, \theta) = h_0(r, t) + \epsilon g(r, t; q)e^{iq(L\cos\alpha)\theta},$$

$$q = \frac{2\pi}{\lambda}$$

Goal: Find growth of g over time for each q

The Equations

Equation for base solution:

$$h_t = -\frac{1}{r}\frac{\partial}{\partial r}\left\{rh^3\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rh_r) + \frac{\tan\alpha}{r}\right) - G\left(\sin\alpha + \cos\alpha h_r\right)\right]\right\}$$

Equation for perturbation:

$$g_t = -\nabla_r \cdot \left\{h_0^3\nabla_r\left[\nabla_r^2 g - \left(\frac{q^2}{r^2\cos^2\alpha} + G\cos\alpha\right)g\right] + 3vg\right\} + \frac{q^2 h_0^3}{r^2\cos^2\alpha}\left[\nabla_r^2 g - \left(\frac{q^2}{r^2\cos^2\alpha} + G\cos\alpha\right)g\right]$$

The Equations

From [2], we define growth rate as:

$$\sigma(t;q) = \frac{1}{g_{max}} \frac{\partial g_{max}}{\partial t},$$

- Growth rate is time dependent
- We only track the growth of ‘the peak’ of the perturbation

We define average growth rate as:

$$\overline{\sigma}(q) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \sigma(t;q)\, \mathrm{dt}$$

[2] J.M. Gomba, J. Diez, R. Gratton, A.G. Gonzalez and L. Kondic. Stability study of a constant-volume thin film flow.

Numerics

Code Breakdown

Baseline: Fortran code performs LSA for constant flux, incline plane

Should provide good benchmark for:

- Small enough curvature
- Early enough times

1. Fortran Code
    a. Iterate on different qs
    b. Calculate base solution as subroutine solving boundary value problem
    c. Then we solve for largest eigenvalue using Rayleigh Quotient which is our growth rate

Code Breakdown

To solve actual experimental setup...

1. Matlab (Funnel, CV)
    a. Precursor Film constructed from initial guess matrix and by solving boundary value problem
    b. Iterate on different qs
    c. Calculate base solution as a function of time (discretizing space) using variable order method (ode15s) and extract perturbation function, g
    d. Output growth rates over time defined as the derivative(g_max)/g_max

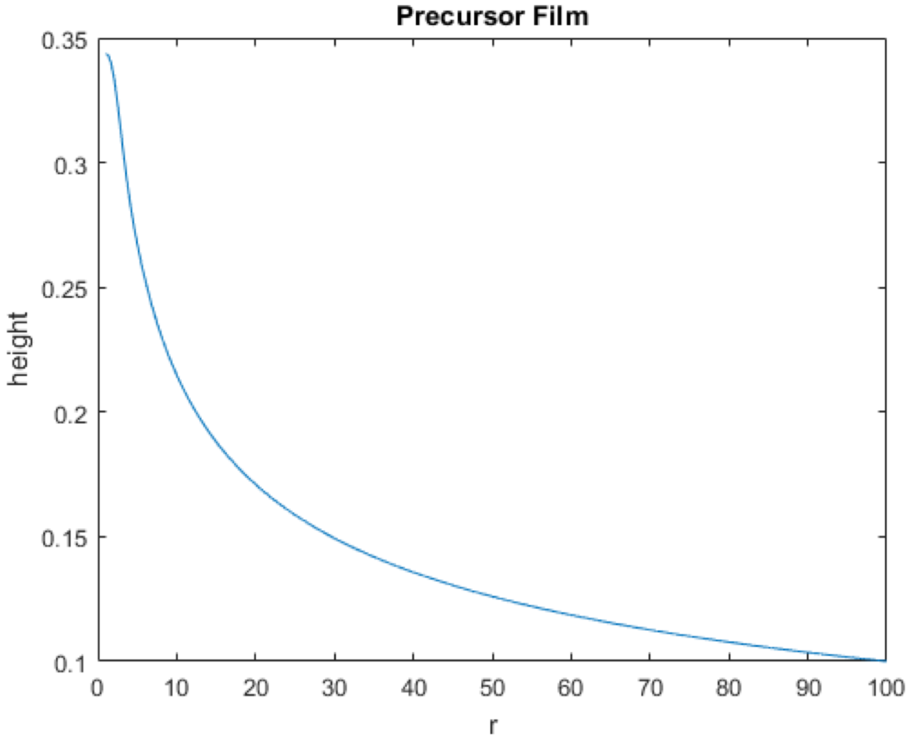
Parameters

Parameters for all numerics,

Base Case	
Angle (degrees)	47
Viscosity (cstokes)	1000
Surface tension (dyn/cm)	20
Density (g/cm^3)	0.97

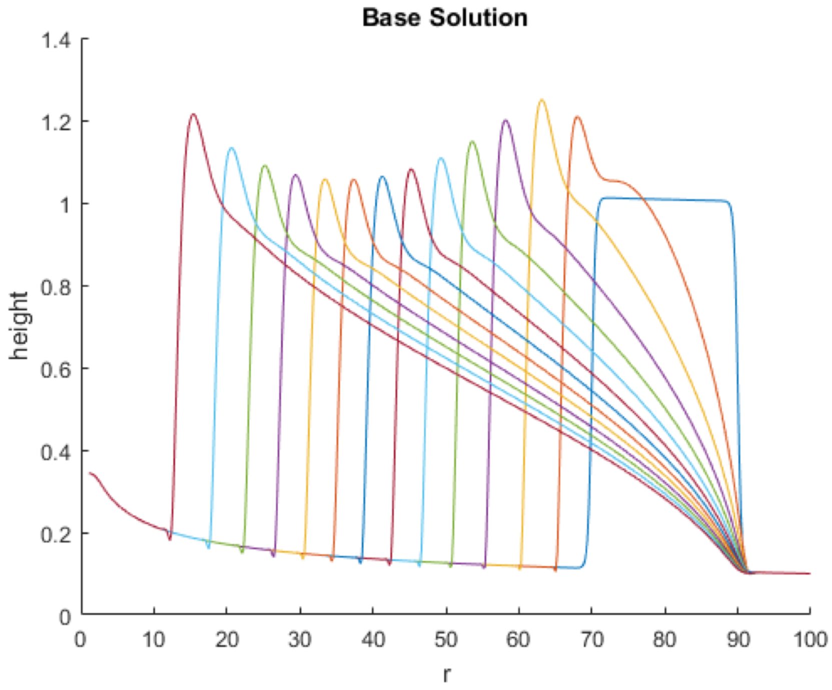
Precursor Film

Given the funnel geometry our precursor film looks like this:



Base solution

Unlike incline plane, we do not have a self similar solution!



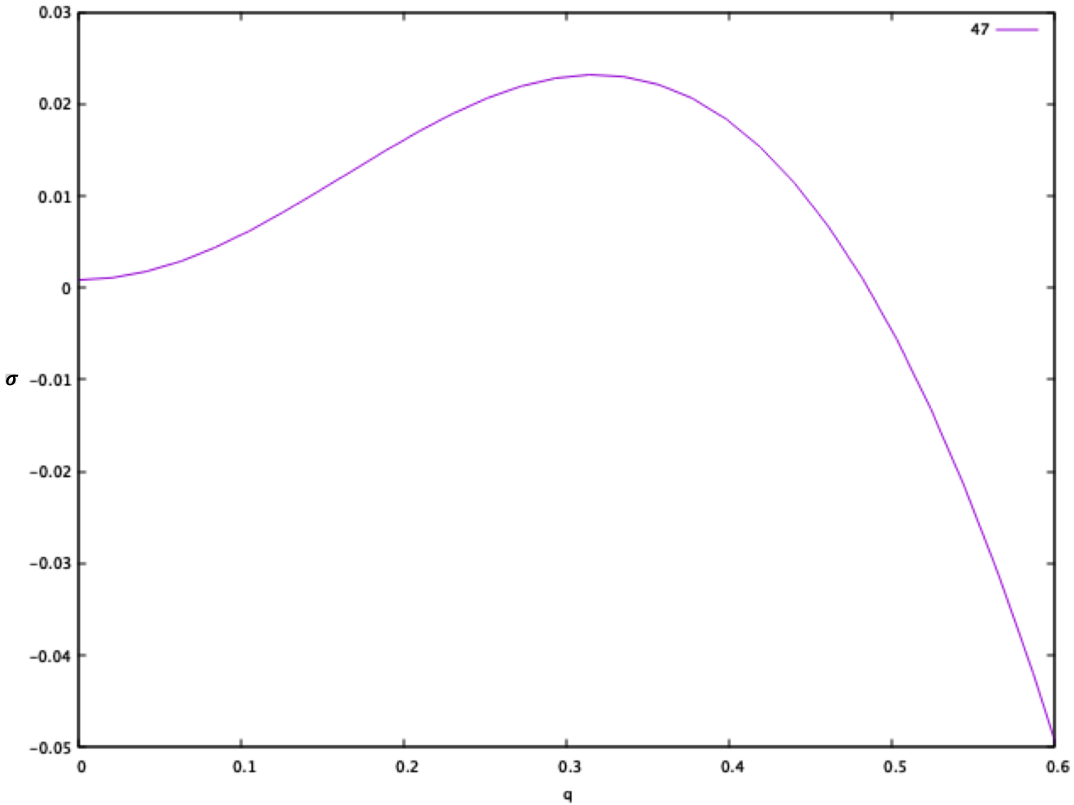
base solution moves in time right to left across the x-axis

Results and Comparison to Experiments

σ vs q, Incline Plane

q_{max}= 0.31

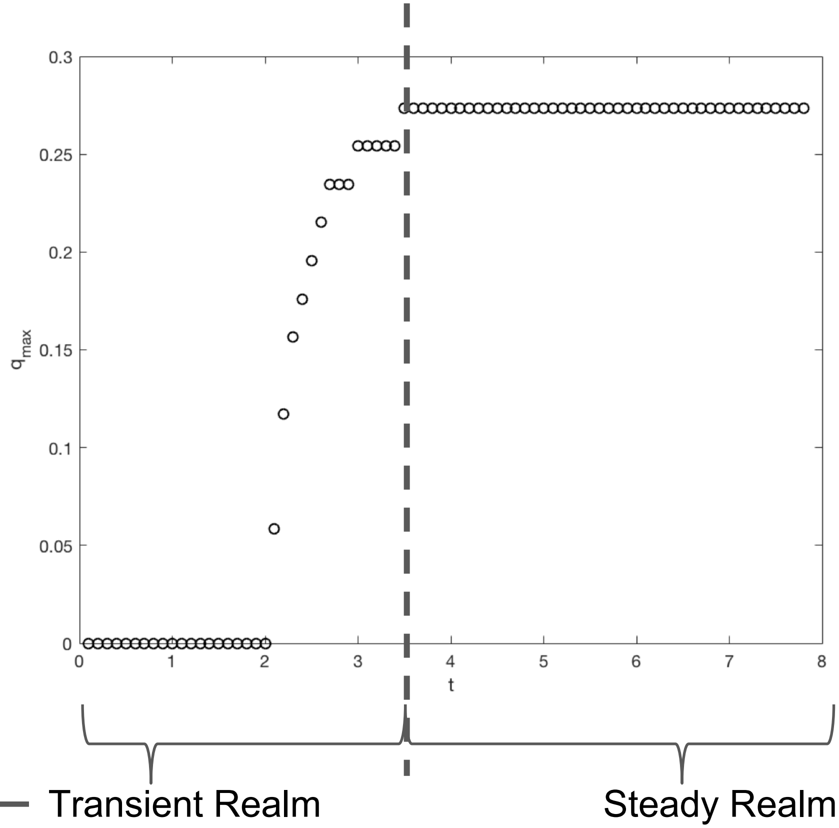
λ_{max}= 2.9 cm



q_{max} vs t, Funnel

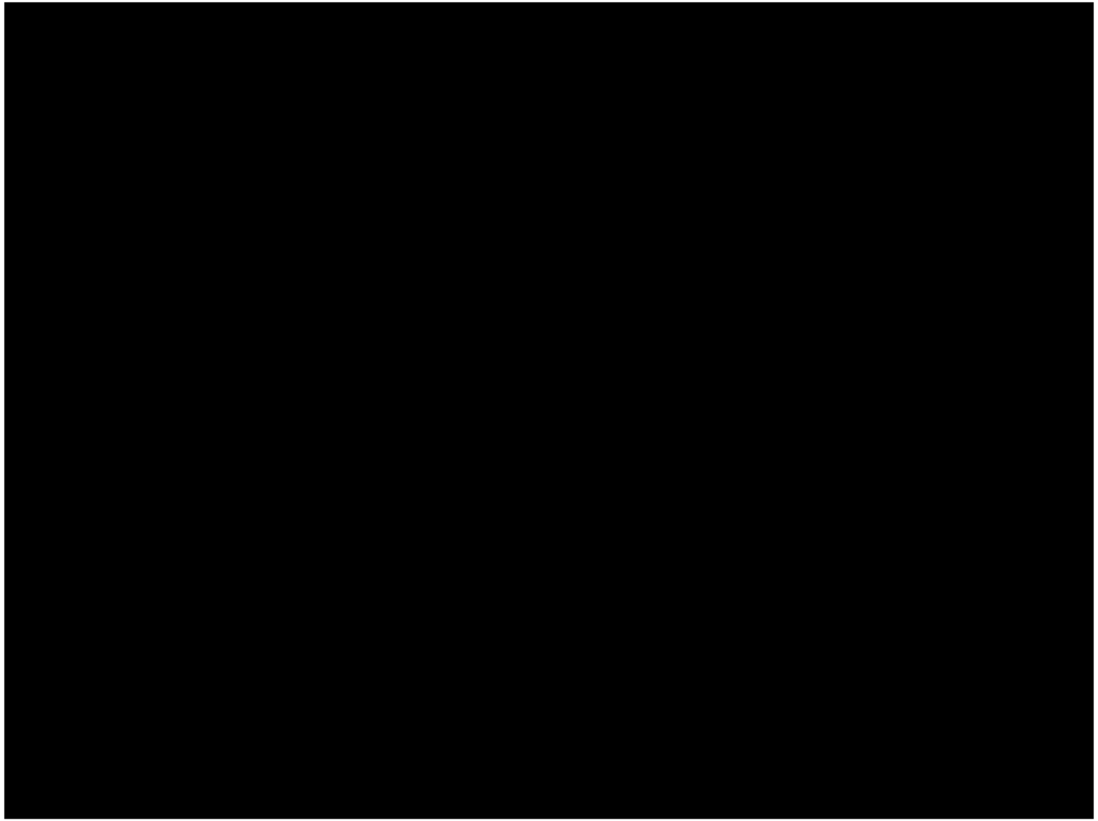
q_{max}= 0.27

λ_{max}= 3.3 cm



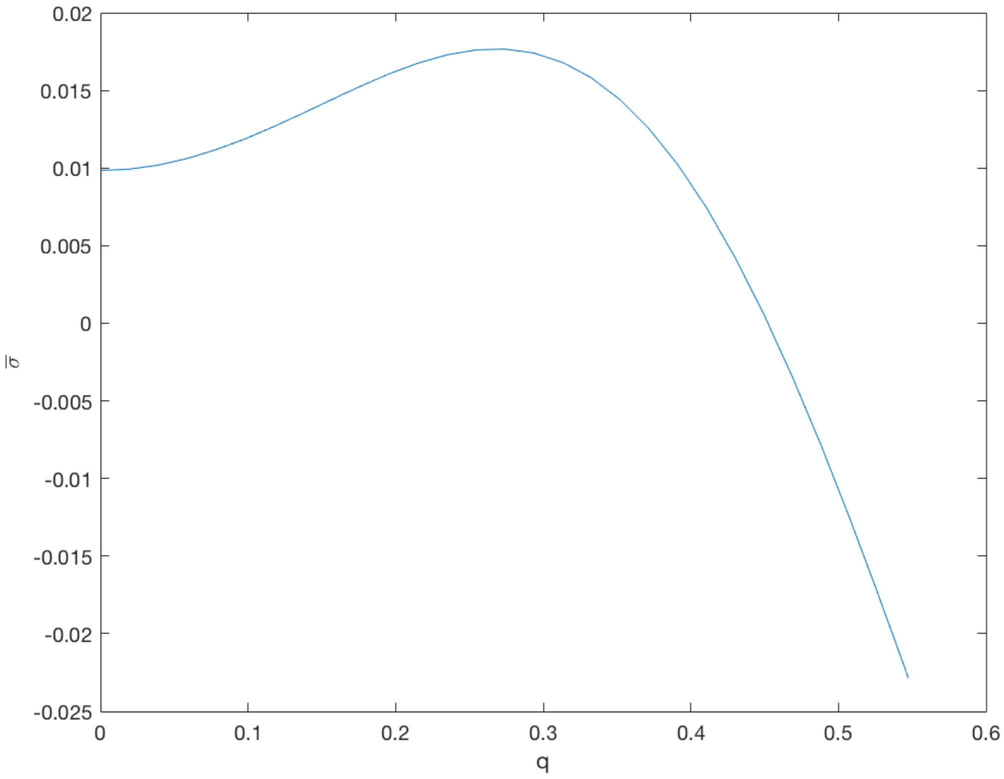
Due to arbitrary initial conditions

$\sigma(t)$ vs q, Funnel



$\bar{\sigma}$ vs q, Funnel

- Average growth rate. Integrated from t_1=4.3 to t_2=7.6



Summary

Summary

- We introduced a **theoretical framework** to study thin film flow in a funnel and perform LSA

- We introduced two **numerical routines**
  - Incline plane constant flux in FORTRAN (benchmark)
  - Funnel constant volume in MATLAB

- We found the separation between the instability fingers for the most unstable mode to be
  - **2.9 cm** in FORTRAN
  - **3.3 cm** in MATLAB

Thank you!

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Appendix