Funnel flow: Self-similar behavior

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Our Problem: Healing Film

- dynamics of a healing film driven by surface tension (inward spreading process of a liquid film to fill a hole)
- Flow driven by surface tension (not gravity, in this particular case)
- Has stable solution

Why?

Reason for flat surface problem:

- Stable solution

Reason for self-similarity:

- Changes our nonlinear PDE into an ODE
- Gives us film thickness near front
- Can be used to compare theoretical and experimental results

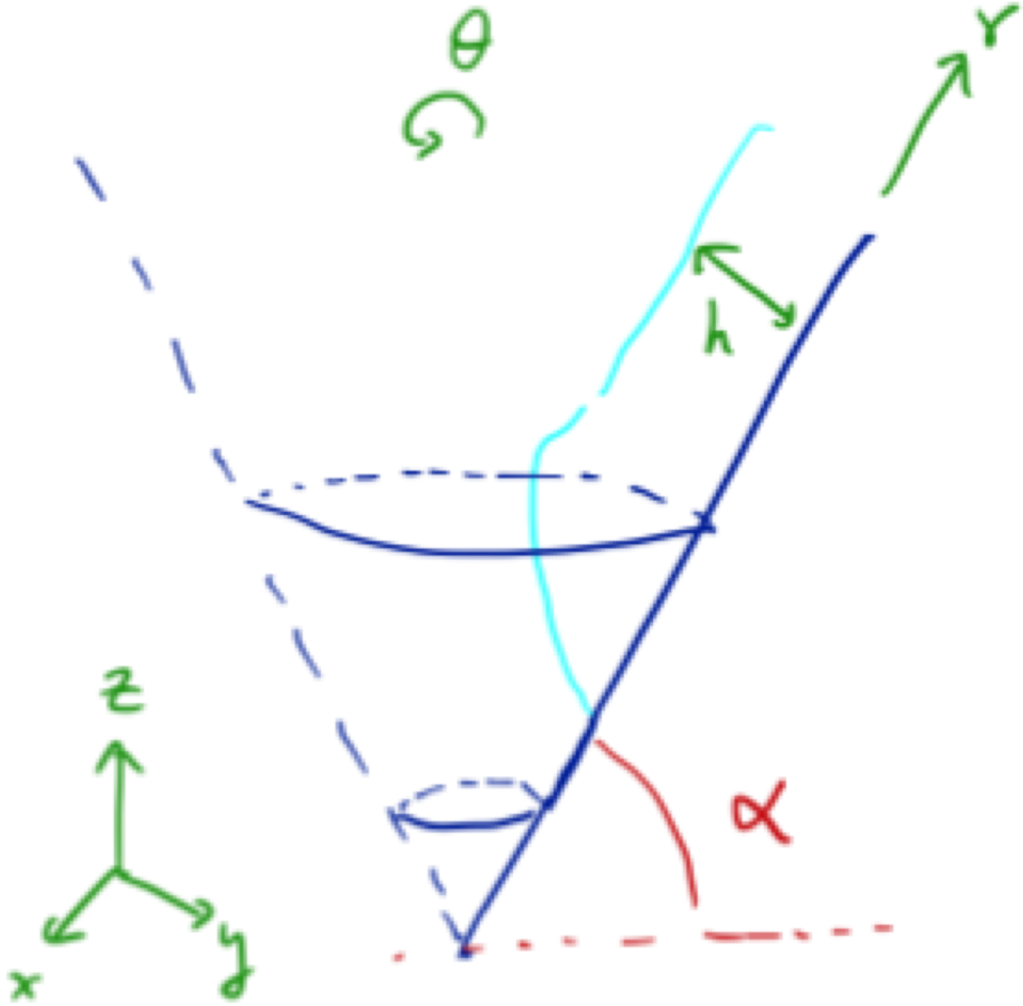
What is a self-similar solution?

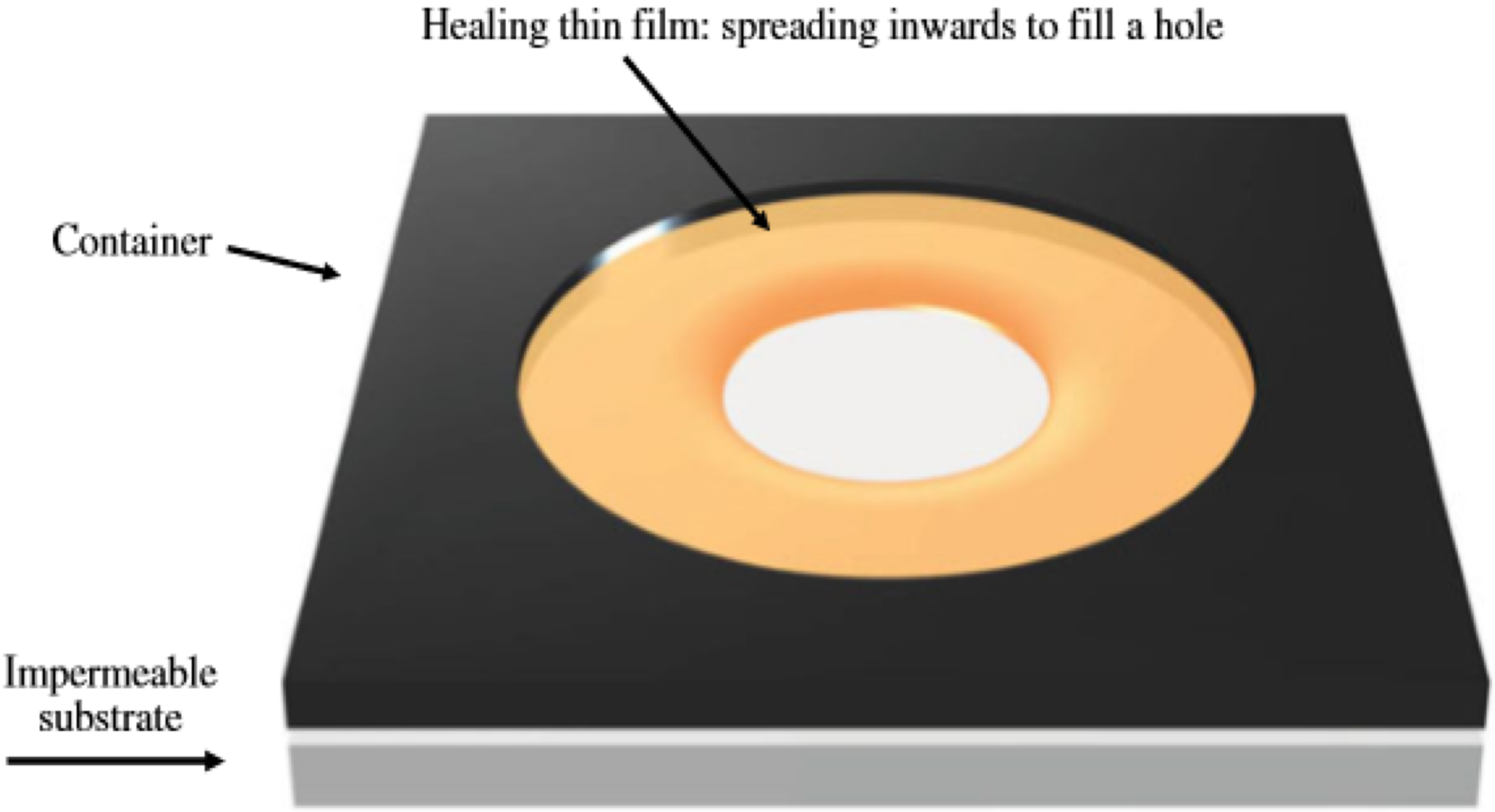
A **self-similar solution** is a form of solution which is similar to itself if the independent and dependent variables are appropriately scaled.

The self-similar solution appears whenever the problem lacks a characteristic length or time scale.

Restrictions/Assumptions

- α=0
- Neglect gravity
- solutions are independent of the θ variable
- Prewetting film


θ
r
h
α
z
x
y


Healing thin film: spreading inwards to fill a hole
Container
Impermeable
substrate

Nondimensionalized Funnel Equations

$$\left(1 \pm \frac{\tan\alpha}{r} h\right) h_t \;=\; -\frac{1}{r}\frac{\partial}{\partial r}\left\{ r h^3 \left[\left(1 \pm \frac{\tan\alpha}{2r} h\right) \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r h_r) \mp \frac{\tan\alpha}{r \pm \tan\alpha h}\right) + \frac{\tan^2\alpha}{r^3} h \right] \right\}$$
$$-\frac{G}{r}\frac{\partial}{\partial r}\left\{ r h^3 \left[-\sin\alpha \left(1 \pm \frac{\tan\alpha}{r} h\right) - \cos\alpha h_r \right] \right\},$$

and

$$h_t = -\frac{1}{r}\frac{\partial}{\partial r}\left\{ r h^3 \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r h_r) + \frac{\tan\alpha}{r}\right) - G\left(\sin\alpha + \cos\alpha h_r\right) \right] \right\},$$

Flat Substrate Equation w/o Gravity

$$\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right)\right)\right) = 0.$$

Similarity

Focus will be on late time dynamics as the hole closes, i.e. as $r_f(t) \to 0.$

We use t_0 to denote the dimensionless healing time, that is, the time for the (circular) front of the air–fluid interface to reach the origin (r = 0).

Self-Similar Transformation

For our equation,
$$\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right)\right)\right) = 0.$$

We will assume that our solution is of the form:

$$h(r,t) = (t_0 - t)^{\alpha} f(\xi), \quad \text{where } \xi \equiv \frac{r}{(t_0 - t)^{\beta}}.$$

We want to end up with an equation that does not explicitly depend on t.

S-S

Substituting our solution into the equation gives us the relationship between the exponents:

$$\alpha = \frac{(4\beta - 1)}{3}.$$

This now gives us the ODE,

$$-\frac{(4\beta - 1)}{3}f + \beta\xi\frac{\mathrm{d}f}{\mathrm{d}\xi} + \frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi f^3\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}f}{\mathrm{d}\xi}\right)\right)\right) = 0,$$

Far-field condition

The similarity solution will not hold everywhere in r and t, but rather, in a region near the apparent contact line, for time sufficiently close to t_0.

Since $\xi \to \infty$ for fixed r as $t \to t0$, the similarity solution is defined on the semi-infinite domain $0 < \xi < \infty$.

The rate of change of film thickness blows up as $(t_0 - t)^{\alpha-1}$

Far-field cont.

However, we expect ∂h/∂t to remain bounded away from the middle of the hole, which requires the first two terms to balance. This gives the far-field 'quasi-stationarity' condition

$$-\frac{(4\beta-1)}{3}f+\beta\xi\frac{\mathrm{d}f}{\mathrm{d}\xi}\to 0 \quad \text{as } \xi\to\infty.$$

Solving this gives us

$$f(\xi)\sim a_1\xi^{(4\beta-1)/(3\beta)} \quad \text{as } \xi\to\infty.$$

Far-Field Asymptotics

Analyse the asymptotic behaviour of the solutions of

$$-\frac{(4\beta-1)}{3}f+\beta\xi\frac{\mathrm{d}f}{\mathrm{d}\xi}+\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi f^3\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}f}{\mathrm{d}\xi}\right)\right)\right)=0,$$

for the special value β = 2/5 in the far field (i.e. as ξ → ∞).

impose the far-field behaviour in the form:

$$f(\xi)=a_1\xi^{1/2}+\bar{f}(\xi)\quad \text{as}\ \xi\to\infty,$$

Then set a1=1 (can be done without loss of generality) and plug into ODE

$$-\frac{1}{5}\tilde{f}+\frac{2}{5}\xi\frac{\mathrm{d}\tilde{f}}{\mathrm{d}\xi}+\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(-\frac{9}{8}\xi^{-1/2}\tilde{f}+\xi^{5/2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}\tilde{f}}{\mathrm{d}\xi}\right)\right]\right)=0,$$

F-F Asymptotics cont.

Can be simplified to

$$\xi^{3/2}\frac{\mathrm{d}^4\tilde{f}}{\mathrm{d}\xi^4}+\frac{7}{2}\xi^{1/2}\frac{\mathrm{d}^3\tilde{f}}{\mathrm{d}\xi^3}+\frac{1}{2}\xi^{-1/2}\frac{\mathrm{d}^2\tilde{f}}{\mathrm{d}\xi^2}+\frac{2}{5}\xi\frac{\mathrm{d}\tilde{f}}{\mathrm{d}\xi}-\frac{1}{5}\tilde{f}=0.$$

WKB Method

Method for finding approximate solutions to differential equations with spatially varying coefficients.

Make substitution: $$\tilde{f} = \mathrm{e}^{-W}.$$

Assumptions: $W'' \ll (W')^2,\ W''' \ll (W')^3$ and $W'''' \ll (W')^4$, where $'$ denotes $\mathrm{d}/\mathrm{d}\xi$

Plug in to get

$$\xi^{3/2}(W')^4 - \tfrac{7}{2}\xi^{1/2}(W')^3 + \tfrac{1}{2}\xi^{-1/2}(W')^2 - \tfrac{2}{5}\xi W' - \tfrac{1}{5} = 0.$$

Far-Field cont.

Another substitution: $$W' = \xi^{-1/6} V.$$

Plug in to get $$\xi^{5/6} V^4 - \tfrac{7}{2} V^3 + \tfrac{1}{2} \xi^{-5/6} V^2 - \tfrac{2}{5} \xi^{5/6} V - \tfrac{1}{5} = 0,$$

Leading order balance: $$\xi^{5/6} V^4 - \tfrac{2}{5} \xi^{5/6} V = 0,$$

Solutions for V: $$V = \left(\tfrac{2}{5}\right)^{1/3}, \left(\tfrac{2}{5}\right)^{1/3} \mathrm{e}^{\pm 2\pi \mathrm{i}/3}.$$

To compute the next-order contribution to V, we write

$$V = \left(\tfrac{2}{5}\right)^{1/3} + \tilde{V}(\xi),$$

cont.

We obtain

$$\overline{V} = \tfrac{8}{6}\xi^{-5/6}\,.$$

Plug back in for W' and solve to get

$$W = \tfrac{6}{5}\left(\tfrac{2}{5}\right)^{1/3}\xi^{5/6} + \tfrac{8}{6}ln(\xi) + O(1)$$

Solving for f to get

$$f(\xi) \sim \xi^{1/2} + \overline{a_2}\xi^{-8/6}\, e^{-(6/5)(2/5)^{1/3}\xi^{5/6}} \text{ as } \xi \to \infty\,,$$

Finding β

Circling back, to complete the self similar solution, a value for Beta must be found.

However, we cannot directly find a ratio that solves for Beta. Therefore, our solution of Beta is a self similar solution of the second kind.

This proved to be somewhat of a challenge, and we ended up using the far field asymptotics to do so.

Finding β using far field asymptotics

Knowing the far field solution of f(here Beta is ⅖) ;

$$f(\xi) \sim \xi^{1/2} + \overline{a_2}\xi^{-8/6}\, e^{-(6/5)(2/5)^{1/3}\xi^{5/6}} \text{ as } \xi \to \infty\,,$$

And also the scaling undertaken for the fluid height h,

$$h(r,t) = (t_0 - t)^{\alpha} f(\xi)\,, \quad \text{where } \xi \equiv \frac{r}{(t_0 - t)^{\beta}}.$$

We can try to find Beta from h!

Finding β using far field asymptotics (cont)

To find Beta, we assume that the fluid front is defined at the height h = 0.01.

Subtracting 0.01 onto the other side of our self-similar formulation, we achieve:

$$0 = (t_0 - t)^{\alpha} * f(\xi) - 0.01$$

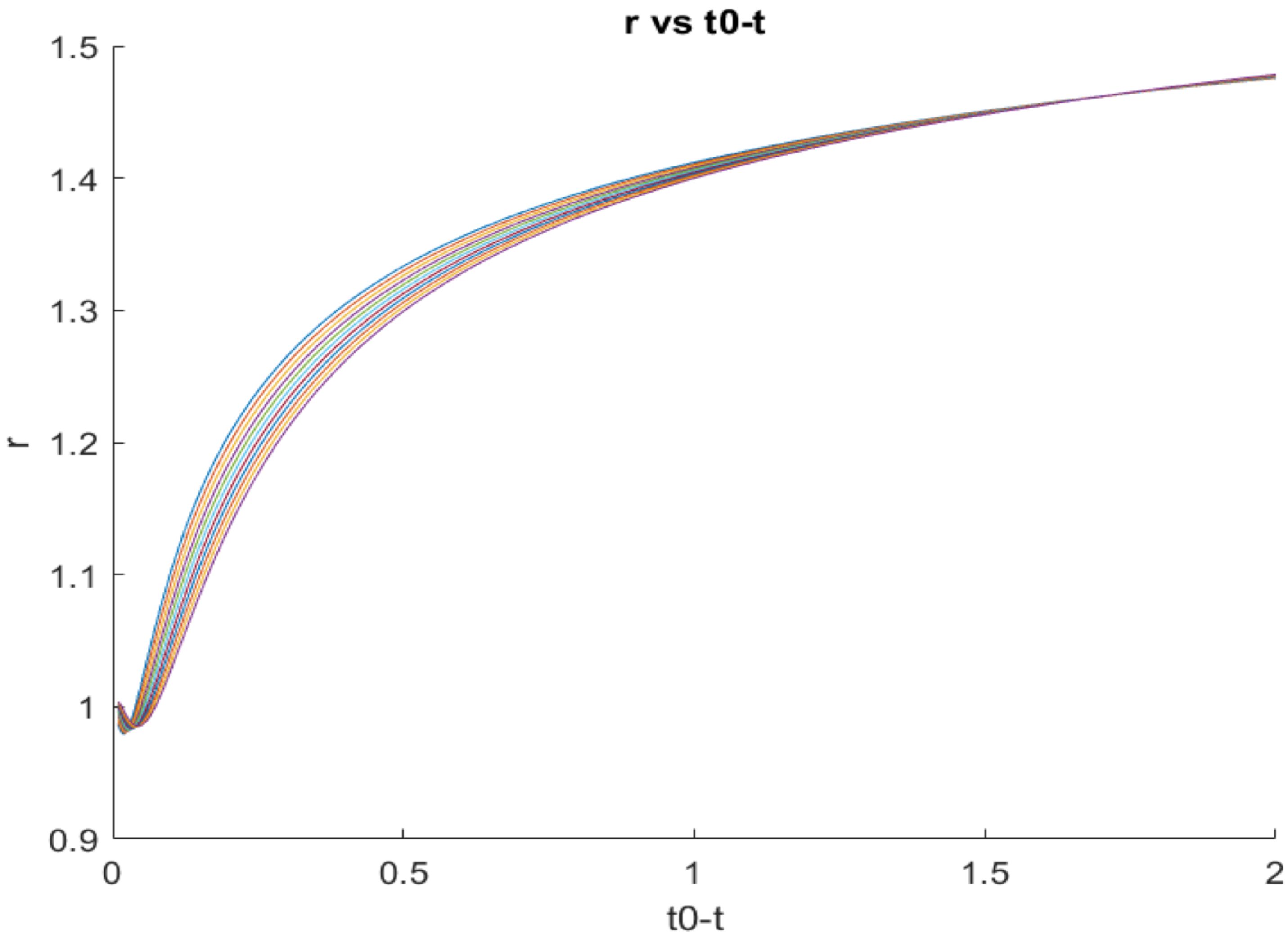
Which we will solve as a root finding problem for the position variable r, at the fluid front.

Finding β using far field asymptotics (cont)

To solve this problem, we need to assume values of Beta. We also do not have established values of a2 or t0, and need to assume values of those as well. This complicates the issue.

We decided to vary all the different values of Beta, a2, and t0, and compare with the other groups results to establish the correct range of Beta. We also used our reference paper as a guide to set the values for Beta and a2.

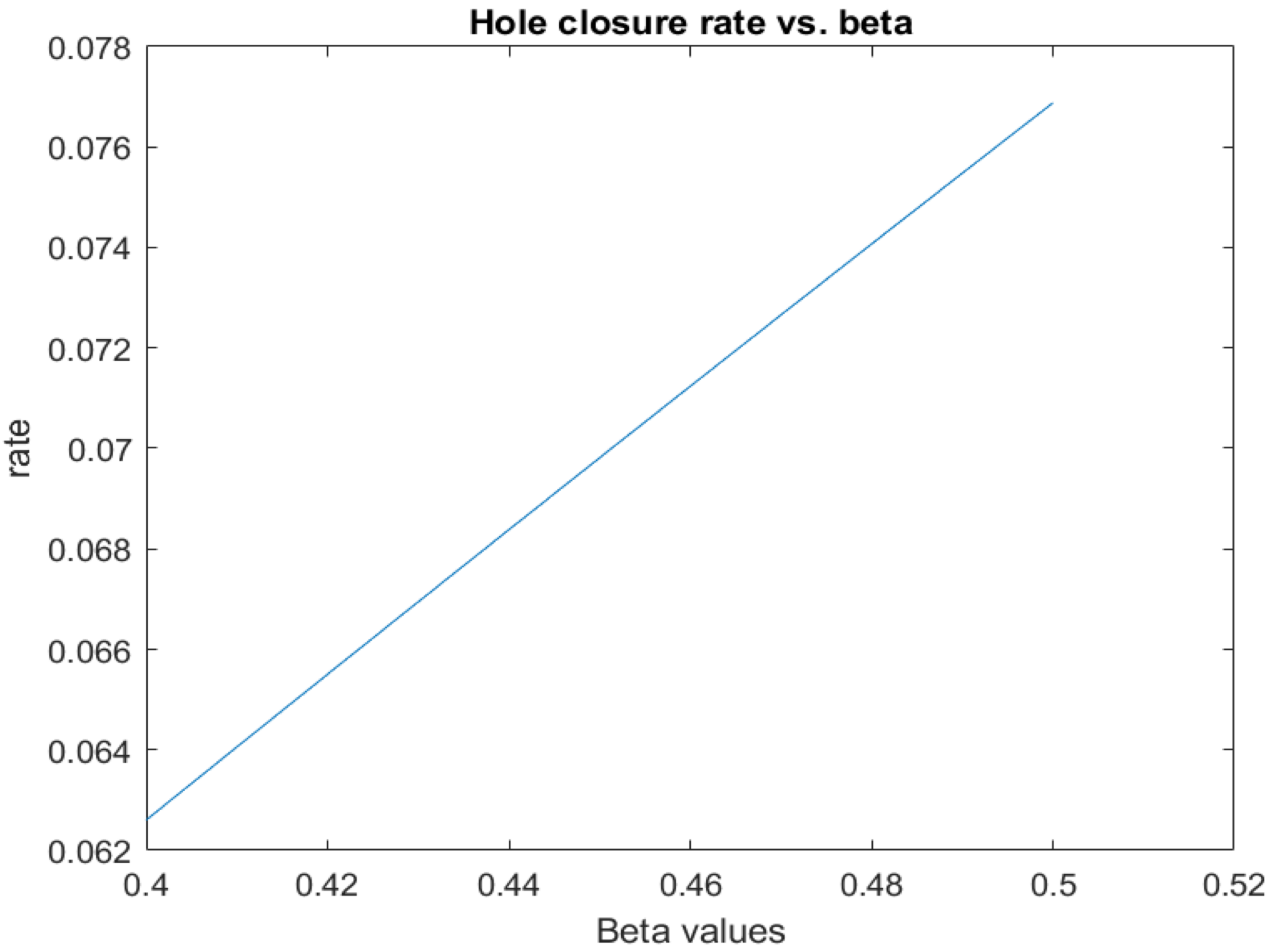
After setting the values of the different parameters, we can obtain a graph of the position of the fluid front as a function of time



r vs t0-t
r
t0-t
1.5
1.4
1.3
1.2
1.1
1
0.9
0
0.5
1
1.5
2

Finding the rate of hole closure

After obtaining a curve like in this graph, we then isolate a more linear part of the curve, and then fit a straight line onto it, the slope of this line is the rate at which the hole closes.



Hole closure rate vs. beta
0.078
0.076
0.074
0.072
0.07
0.068
0.066
0.064
0.062
rate
0.4
0.42
0.44
0.46
0.48
0.5
0.52
Beta values

General observations

Reducing t0 causes the rate of closure to vary more greatly as beta changes, and increasing t0 causes the rate of closure to vary less as beta changes.

Changing the magnitude of a2(only negative values work) does not cause the rate of closure to change greatly, but reducing the magnitude of a2 will increase the overall rate, and increasing it will decrease the base rate.

Changing beta tends to have less of an overall effect on results, indicating that a range of values might work. It will likely be necessary to use data fitting with experimental results to produce an exact value of beta.

Vs. Other Group Results

It is very difficult to directly compare our results to the other group results, as our results are non-dimensional and the rates obtained by the experimental group are dimensional. In analyzing our non-dimensionalization we could not find some values from the experiment.

$$\tilde{t}_c = \frac{3\mu\tilde{r}_0^4}{\gamma\tilde{h}_0^3}$$

In this scaling the characteristic time is going to be rather large.

Vs. Other Group Results(cont.)

Therefore our characteristic time is very large, and any non-dimensionalized times will be very small as a result. This means any rates we find should be much larger in comparison to the experimental results(~0.04-0.05 cm/s), based on values of a2 and t0 we can set (a2 and t0 should be small in magnitude).

$$\frac{\Delta r}{\Delta t} = \frac{t_c}{r_0}\frac{\Delta r_p}{\Delta t_p}$$

In the Future

- Investigate near-field asymptotics
- Further comparison between group results

References

[1] Z. Zheng, Healing Capillary Films, J. Fluid Mech. (2018), vol. 838, pp. 404–434. Cambridge University, 2018.

[2] Te Sheng Lin, Funnel Problem, February 27, 2019.