Modeling of Thin Film Flow Driven by Surface Acoustic Wave and Gravity

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Abstract

Surface acoustic wave refers to a phenomenon of fluid flow due to the vibration of the surface. This is not a trivial behavior, especially when we observe how fluid behaves in various situations. When the drop of fluid is placed on the surface of solid, the vibration of the surface obviously affects the fluid. Figure 1 visualized this process. When the vibration is induced to the solid, the vibration propagates through the surface in a form of transverse wave(Rayleigh wave). When this wave enters the region of liquid, the momentum of the wave is absorbed. This absorbed momentum converts into longitudinal pressure wave inside the liquid. At a certain frequency, usually at MHz range, this causes a movement of fluid.

These process seems intuitive. However, there are many interesting behaviors that need further explanations. At a certain condition, the fluid actually flows toward the source of the wave. Also, the speed of flow of fluid depends on the frequency, viscosity, density, and many other variables. Because different types of fluid has different characteristics, we can observe the one fluid run over other fluid. These phenomena is counter-intuitive, and mathematical modeling can provide good explanations for those weird behaviors. In this paper, governing equation of fluid under gravitational force and acoustic force is derived, and for simplified one-dimensional equations, numerical simulation is done using Finite Difference Method.

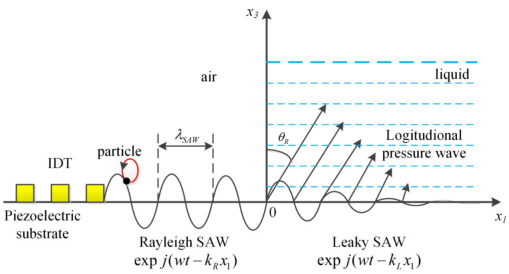

Figure 1: isualization of Surface Acoustic Forcing[1]

1 Introduction

As we are dealing with fluid, the basic governing equation will be the incompressible Navier-Stokes equation[2].

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + c\nabla^2 \mathbf{u} + \mathbf{F} \qquad (1)$$

This equation is a fluid version of Newton's second law. The left side can be summarized as a product of fluid density and time derivative of fluid velocity, which is exactly the product of mass and acceleration. The right side of the equation is collection of forces. The pressure gradient, viscous force, and any external forces will be represented in F. This interpretation gives an idea why this equation governs the fluid. This is the reason why the fluid that governs by this equation is called Newtonian fluid.

This equation describes arbitrary flow of fluid with arbitrary external force, so we need to enforce our conditions. The two main external forces we are considering are gravity and acoustic forcing, since our goal is simulation of fluid under gravity and surface acoustic waves. We derive a governing equation with gravity only, and then add acoustic forcing to it.

$$\begin{aligned}\frac{\partial \overline{\mathbf{u}}}{\partial \overline{t}} + (\overline{\mathbf{u}} \cdot \overline{\nabla})\overline{\mathbf{u}} &= \frac{1}{\rho}\overline{\nabla}\overline{p} + \frac{\mu}{\rho}\overline{\nabla}^2\overline{\mathbf{u}} \\ &+(gsin(\alpha))\hat{i} - (gcos(\alpha))\hat{k}\end{aligned} \qquad (2)$$

$$\begin{aligned}\frac{\partial \overline{\mathbf{u}}}{\partial \overline{t}} + (\overline{\mathbf{u}} \cdot \overline{\nabla})\overline{\mathbf{u}} &= \frac{1}{\rho}\overline{\nabla}\overline{p} + \frac{\mu}{\rho}\overline{\nabla}^2\overline{\mathbf{u}} \\ +(gsin(\alpha) - (1+{\alpha_1}^2)A^2\omega^2\overline{k}_i e^{2(\overline{k}_i\overline{x}+\alpha_1\overline{k}_i\overline{z})})\hat{i} \\ -(gcos(\alpha) + (1+{\alpha_1}^2)A^2\omega^2\overline{k}_i e^{2(\overline{k}_i\overline{x}+\alpha_1\overline{k}_i\overline{z})})\hat{k}\end{aligned} \qquad (3)$$

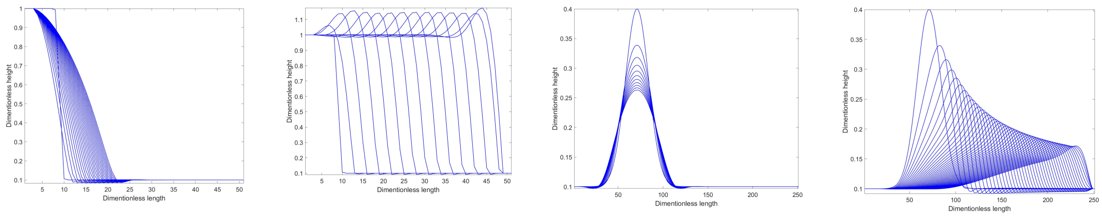

Figure 2: Numerical simulation for equation (5). First and third plots shows result with α=0, second and forth plots plots shows result with α=80. Grid size 0.5. Time step 0.0001

$$k_i = x_c \left[(k_i^{oil} - k_i^{air})(1 - e^{-\frac{x_c(h-b)}{\lambda}}) + k_i^{air} \right] \quad (4)$$

The equation (2) governs the fluid flow under gravitational force, meaning fluid over an inclined surface[3]. Equation (3) governs the fluid under gravity and acoustic forcing.[1]

Now, these equation is three-dimensional, and indeed the fluid flow is a three-dimensional phenomenon. However, due to the complexity of these equations, we simplified these equations with various methods.

$$\begin{aligned} \frac{\partial h}{\partial t} &= Dcos(\alpha)\nabla \cdot [h^3 \nabla h] - \nabla \cdot [h^3 \nabla^3 h] \\ &\quad -sin(\alpha)\frac{\partial h^3}{\partial x} \end{aligned} \quad (5)$$

$$\begin{aligned} \frac{\partial h}{\partial t} &= Dcos(\alpha)\nabla \cdot [h^3 \nabla h] \\ +C\nabla \cdot [h^3 e^{2k_i x}] &- \nabla \cdot [h^3 \nabla^3 h] - sin(\alpha)\frac{\partial h^3}{\partial x} \end{aligned} \quad (6)$$

The equation (5) is simplified one-dimensional equation for the thin film flow under gravity, and equation (6) is simplified equation for gravity and acoustic forcing[4]. The detailed derivation is attached as a separate file.

2 Methods

The method we use is an explicit Finite Difference Method. It is the most simplest, and low-complex method to solve partial differential equations. The FDM scheme directly comes from the definition of derivative.

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad (7)$$

In equation (7), when the infinitesimal h is replaced with the finite step size dx, it becomes a first-order FDM scheme. Now, this differentiating method is called forward difference, because it is using the current step to calculate the state on one time step forward. This method has an error of O(n), which is far from efficient. With a small exchange with computational cost, we can use method called Central Difference method.

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} \quad (8)$$

The equation (8) uses one point before to calculate the derivative[5]. This method has $O(n^2)$ error, which is slightly better than forward differencing. We can use this to derive discritizing schemes for higher derivatives. Our equation contains at most fourth derivatives, so a combination of these schemes give us a completely discritized equation[2].

$$u_j' = \frac{u_{j+1} - u_{j-1}}{2dx} + O(dx^2) \quad (9)$$

$$u_j'' = \frac{u_{j+1} - 2u_j + u_{j-1}}{dx^2} + O(dx^2) \quad (10)$$

$$u_j''' = \frac{u_{j+2} - 2u_{j+1} + 2u_{j-1} - u_{j-2}}{2dx^3} + O(dx^2) \quad (11)$$

$$u_j'''' = \frac{u_{j+2} - 4u_{j+1} + 6u_j - 4u_{j-1} + u_{j-2}}{dx^4} + O(dx^2) \quad (12)$$

With these scheme, our fully discretized equation is,

$$\begin{aligned} u_j^{i+1} &= [Dcos(\alpha)[u^{3'}u^{'} + u^3 u^{''}] - [u^{3'}u^{'''} + u^3 u^{''''}] \\ &\quad -[sin(\alpha)u^{3'}]]dt + u_j^i \end{aligned} \quad (13)$$

This specific method is called the Forward Time Centered Space(FTCS) method. As its name suggests, this method uses central differencing for spatial domain, and uses forward difference for time step. Therefore, this method is a first-order method in time.

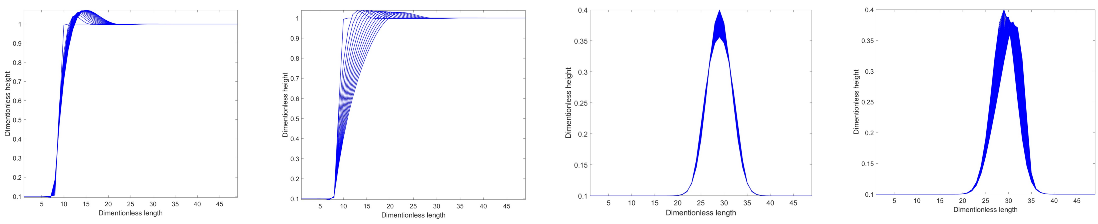

Figure 3: Numerical simulation for equation (6). First and third plots shows result with α=0, second and forth plots plots shows result with α=80. Grid size 0.5. Time step 0.0001

3 Results

The simulation is done for two different external forces, gravity and surface acoustic wave. Each simulation is done for two different initial profiles, hyperbolic tangent and bell curve. Hyperbolic tangent is used to simulate traditional fluid flow, and bell curve is used to simulate a liquid droplet.

Figure 2 shows the simulation result of equation 3, for two different initial profiles. When the angle of incline is zero(first and third plots), there was no horizontal movement, and fluid just slowly flattened over time. When the angle of incline is not zero(second and forth plots), the rapid forward movement is observed in both initial conditions. These results do fit the experimental results and our physical intuitions[3].

Figure 3 shows the simulation result of equation 4, for the same initial profiles. Because there is continuous acoustic forcing, the liquid does move forward even when the angle of incline is zero. The increase in angle of incline only increases the speed of movement. However, the result is not as significant as other simulations.

4 Discussion

There were many obstacles to building a working numerical method scheme. First of all, our governing equation was not correct. The original governing equation, equation (2), (3) is reduced to equation (5) and (6). However, these simplified equation seems not quite right. The gravitation-only case was relatively well simulated, and this is obvious because the simplified equation was directly derived from existing research[3]. It only used to check if our numerical scheme does work correctly. Indeed, our FTCS method works very well for equation (5), as you can see from figure 2. In both hyperbolic tangent and bell curve initial profile, both do show exactly the same result shown in paper[3]. This confirms the our approach to the solution was not wrong.

However, we have many problems during simulating equation (6). This equation actually carries an additional term, but that term causes the solution to converge to zero only after a few time steps. I personally can't find any flaw from the derivation of the simplified equation, and my only assumption is that it is caused by wrong non-dimensionalization. There are two reasons why we think this is a problem. One of them is that the simulation of the equation actually does work, but just way too slowly. From figure 3, you can see that for both initial profiles, the movement is way too slow compared to figure 2. This suggests that some of the parameters are not properly normalized. The second reason is about the original equation. The simplified equation actually contains the fifth term, which was multiplied by $\frac{3}{8k_i^3}$, where k_i is defined as equation (4). The k_i term is huge, as the value of k_{oil} is -1000. Since the fifth term includes a cubic of k_i in the denominator, the term disappears immediately. With this fifth term, our group was not able to build a well-behaving simulation, therefore we just ignored it.

Also the stability analysis was skipped. Our purpose was to build a working mathematical model first, and then do additional analysis on it. However, we were suffering from building a working model in the first place, therefore no further analysis was done. Although, we did observe that this explicit FTCS scheme is very unstable, and requires an incredible amount of time step in order to show stable results. We thought this is caused by the high-order derivative term, but at the end we learn that the traditional CFD stability condition does not apply to our equation at the first place. A further analysis on stability also needs to be done in order to ensure the validity of our solution.

5 Conclusions

Based on simulation results, our group was able to build and check the validity of the FDM method for equation (3). The result matches well with real world experiment and numerical simulation done in other papers. However, we failed to build a good simulation for equation (4), and despite the long investigation we can't find out the cause.

References

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