A Free-Streamline Model for a Liquid Film in a Fast Flow

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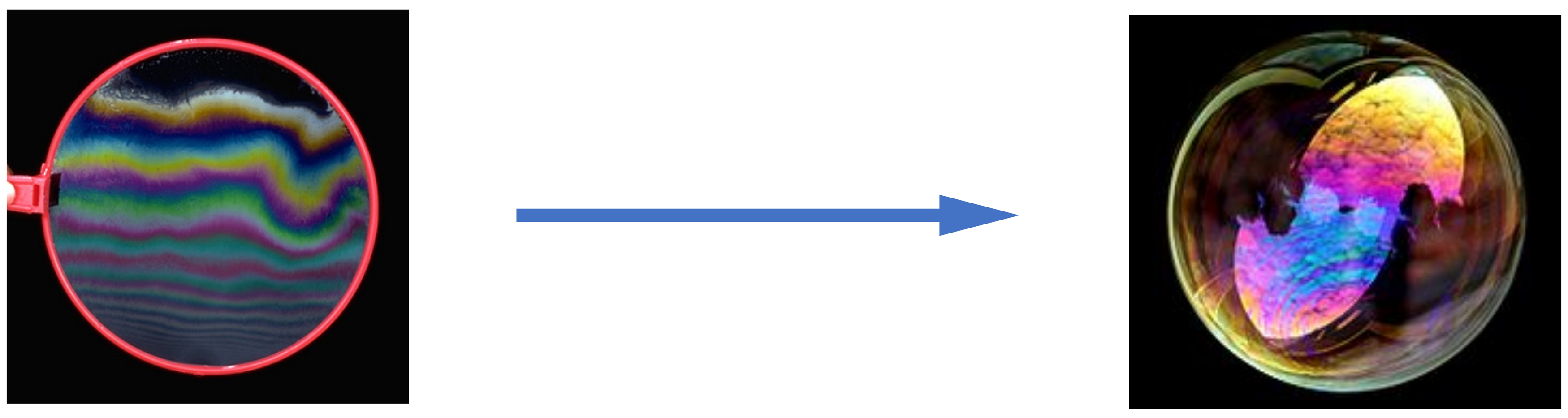
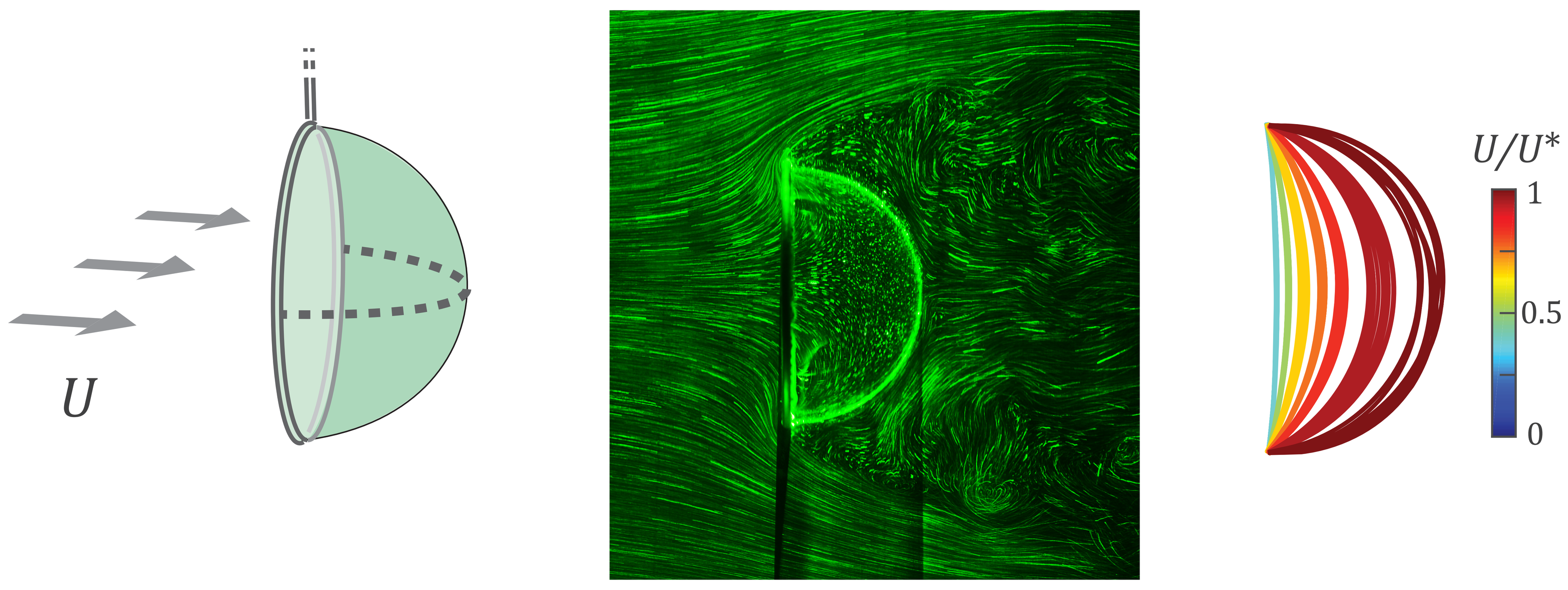
Motivation

- How does a blown film become a bubble?
- What are the hydrodynamic events that trigger necking down and pinching off?

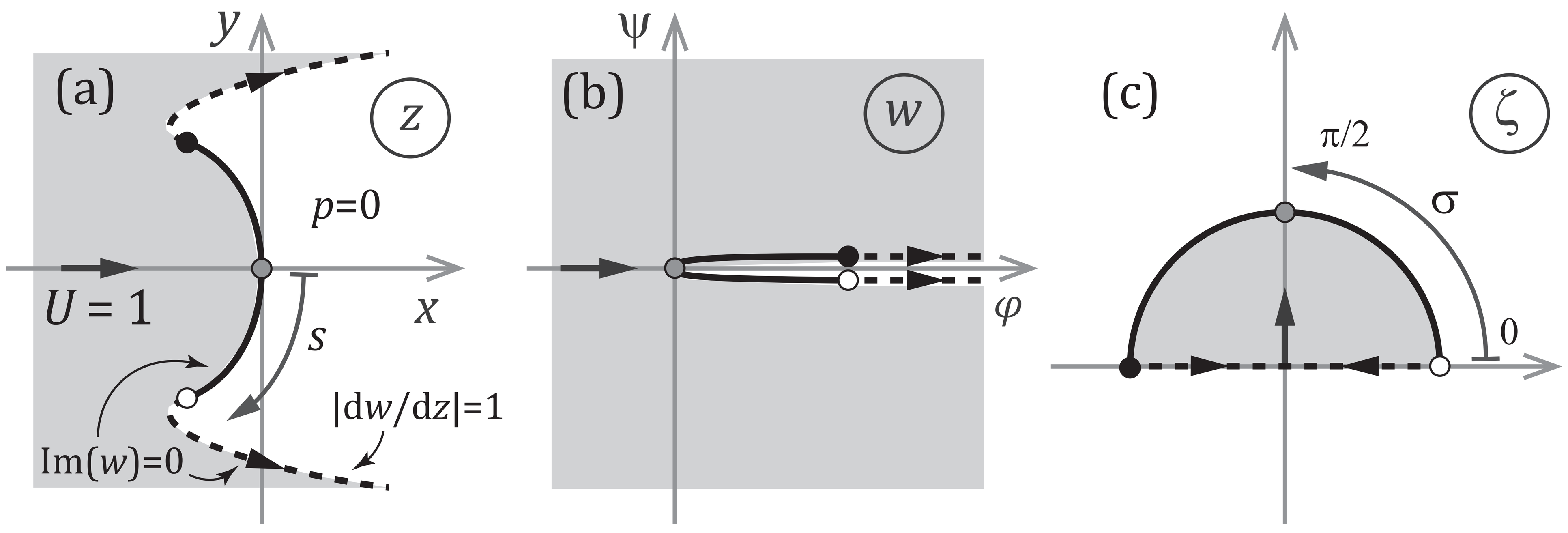


Experimental setup

- Consider a liquid-liquid system: an oil film in a water tunnel.
- The film inflates as the flow speed U is increased, and ruptures at a critical speed U^*.



Free-streamline theory: a 1D film in a 2D flow



$$\frac{d\theta}{d\sigma} = -K\eta \sinh[\tau(\sigma)]\sin 2\sigma, \quad \theta(\pi/2) = -\pi/2,$$
$$\frac{dz}{d\sigma} = -\frac{K}{2} e^{i\theta} e^{-\tau} \sin 2\sigma, \quad \mathrm{Im}\,[z(0)] = -1,$$
$$\tau = \mathcal{H}[\theta].$$

- Flow (gray) is assumed to be inviscid and irrotational.
- Free streamlines (dashed curves) have unknown shape, and shed tangentially from the pinning points.
- The wake (white) is assumed to have uniform pressure.
- Film shape (black curve) is determined by the balance of hydrodynamic pressure and surface tension.
- The potential flow is found using conformal mapping.

- Let $\Omega = i \log(dw/dz)$, where $w(z)$ is the velocity potential.
- Define $\Omega = \theta(\sigma) + i\,\tau(\sigma)$ on the film, and $w(\zeta) = (K/8)(\zeta + \zeta^{-1})^2$.
- $\Omega \in \mathbb{R}$ for $\zeta \in \mathbb{R}$ (free streamline). Schwarz reflection principle implies that Ω may be extended to the unit disc in the ζ-plane:

$$\Omega(\zeta) = \textstyle\sum_k a_k \zeta^k \quad \text{where} \quad a_k \in \mathbb{R}.$$

Thus, $\tau = \mathcal{H}[\theta]$, where $\mathcal{H}$ is the Hilbert transform.

- Governing equation is obtained by combining

$p = 2\gamma\kappa$ **(Young-Laplace equation)**

$p + \frac{1}{2}\rho|\boldsymbol{u}|^2 = \frac{1}{2}\rho U^2$ **(Bernoulli's Law)**

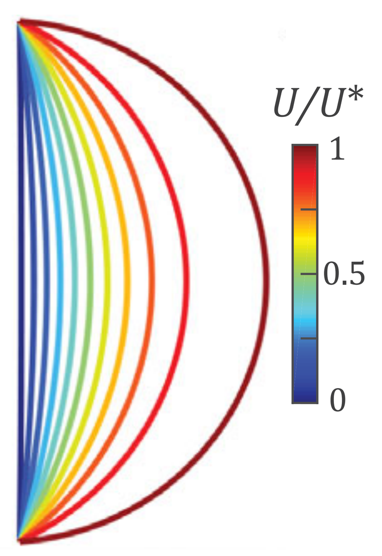
where p is the hydrodynamic pressure, γ the surface tension, κ the film curvature, ρ the fluid density, and $\boldsymbol{u}$ the flow velocity.

- The Weber number is $\eta = \rho U^2 R / 4\gamma$, where R is the ring radius.
- Equation is solved numerically using a Broyden method, with η as the continuation parameter.

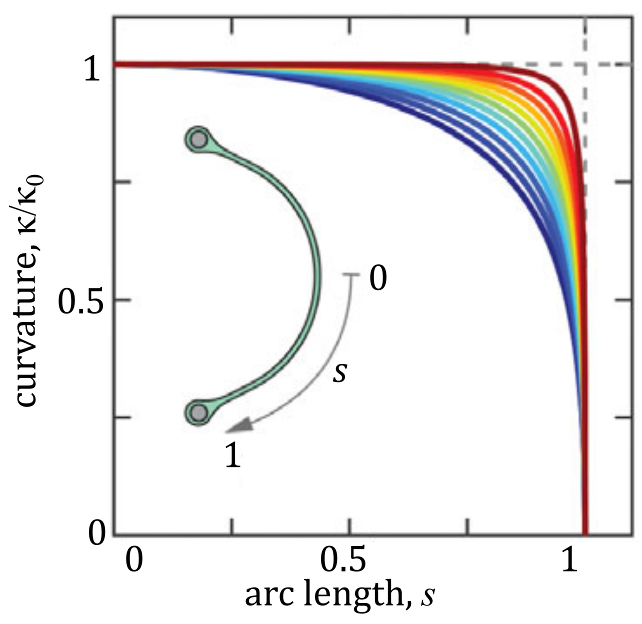
Results

- Highly inflated films have nearly uniform curvature, except near the pinning points.

Predicted film shapes

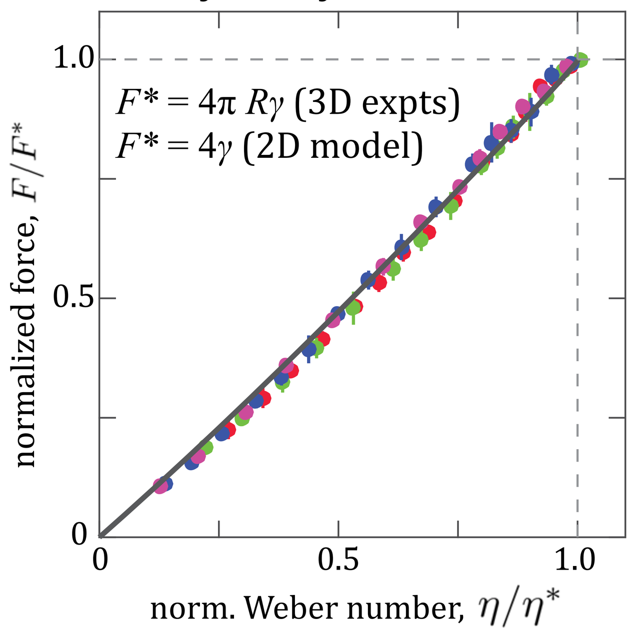


Film curvature

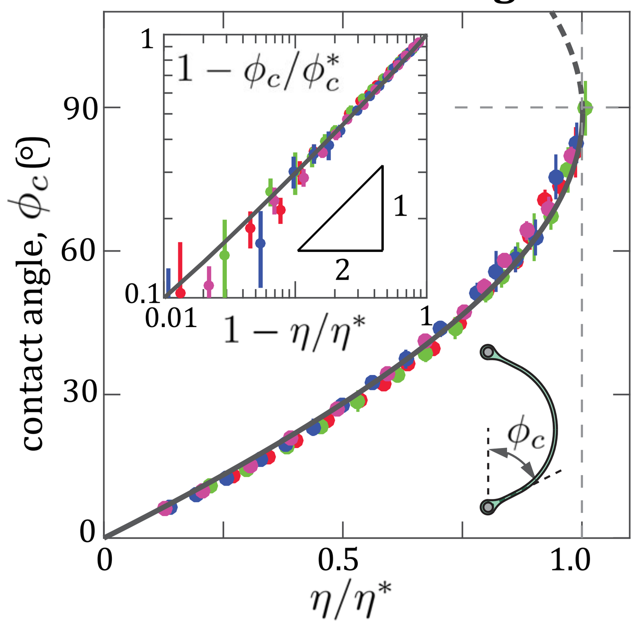


- Theory exhibits favorable agreement with experiment.

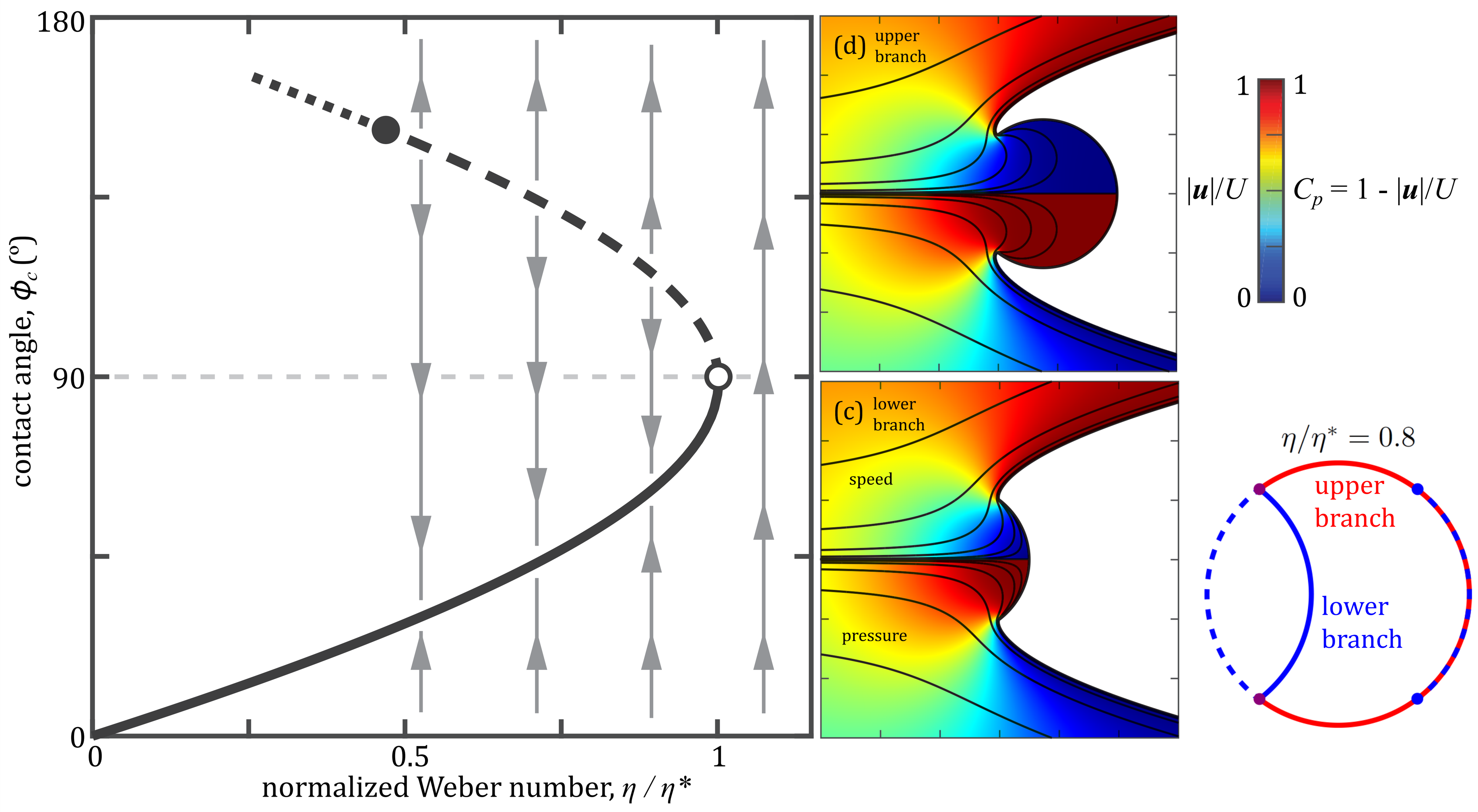
Hydrodynamic force



Film contact angle



Model prediction: over-inflated equilibrium shapes



- **Stable shape**: Inward deformations lower the Laplace pressure due to the **decrease** in curvature, so the external pressure dominates and restores the shape.
- **Unstable shape**: Inward perturbations **increase** curvature, leading to further collapse of the film.

Conclusions

- Rupture results from a maximal surface tension force that can resist imposed fluid pressures.
- Rupture is associated with the critical point of a saddle-node bifurcation.
- Future work: what conditions determine the different fates of rupture vs. birthing of a bubble?

References & Acknowledgments

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- Reference: Ganedi *et al.*, "Equilibrium states and their stability for liquid films in fast flows," *Physical Review Letters* **121**, 094501 (2018).